université PARIS-SACLAY

# Right-invariant sub-Riemannian geometry in the Banach setting and applications to large deformations models in shape analysis

*Géométrie sous-Riemannienne invariante à droite sur les espaces de Banach et applications aux modèles de grandes déformations pour l'analyse de formes*

**Thèse de doctorat de l'université Paris-Saclay)**

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THESE DE DOCTORAT


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**ÉCOLE DOCTORALE**
de mathématiques
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**Titre:** Géométrie sous-Riemannienne invariante à droite sur les espaces de Banach et applications aux modèles de larges déformations pour l'analyse de formes

**Mots clés:** géométrie sous-Riemannienne en dimension infinie, contrôle optimal, groupe de difféomorphismes, groupes de Lie, analyse des formes, anatomie computationnelle

**Résumé:** Cette thèse est dédiée à l'étude de structures sous-Riemanniennes fortes et invariantes à droites sur des groupes de dimension infinie et les espaces de formes. Elle étend en particulier le cadre géométrique des grandes déformations, qui décrit les espaces de formes comme des variétés de Banach sur lesquelles agissent un groupe de diffeomorphismes. Ici nous élargissons ce cadre en autorisant l'action d'autres groupes de déformations sur les formes, ce qui donne lieu à de nouveaux problèmes d'appariement. Nous nous plaçons dans le cadres des Banach half-Lie groups et étudions d'abord les propriétés des structures sous-Riemanniennes fortes et invariantes à droite sur ces groupes. Sous certaines hypothèses, il est alors possible d'établir des résultats de complétudes pour ces métriques, et nous démontrons en particulier l'existence global du flot Hamiltonien associé. Ensuite, nous définissons des conditions de régularité pour l'action des half-Lie groups sur les espaces de formes, ce qui permet alors d'induire sur ces espaces des structures sous-Riemanniennes. Ce cadre nous permet alors de formuler différents problèmes variationnels pour l'appariement de formes. Plusieurs applications sont alors présentées. Dans le chapitre 7, nous proposons une approche multi-échelles, et nous explorons tout particulièrement des manières de coupler les actions des difféomorphismes et de groupes de Lie de dimension finie, tels que les isométries ou scalings. Les chapitres 8 et 9 s'intéressent ensuite à l'anisotropie des formes, que l'on décrit par des métriques de l'espace ambiant. Nous définissons alors différentes actions de groupes sur ces métriques permettant de transporter les caractéristiques anisotropes des formes.

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**Keywords:** sub-Remannian geometry in infinite dimensions, optimal control, diffeomorphism groups, Lie groups, shape analysis, computational anatomy

**Abstract:** This thesis is dedicated to the study of strong right-invariant sub-Riemannian geometry on infinite dimensional groups and shape spaces. In particular, it extends the geometric framework of large deformations, which models shape spaces as Banach manifolds acted upon by groups of diffeomorphisms. We broaden this setting by allowing other groups of deformations to act on shapes, giving rise to new matching problems. We consider the setting of half-Lie groups and we first study theoretical properties of strong right-invariant sub-Riemannian structures on these groups. Under suitable regularity conditions, we establish completeness results for such structures and prove global existence of their associated Hamiltonian flows. We then derive regularity conditions on the action of half-Lie groups on the shape spaces that make it possible to induce corresponding right-invariant metrics. This framework enables to define general variational matching problems for shape analysis using the sub-Riemannian energy. Several applications are presented. In chapter 7, we propose a multiscale approach for performing registration, and we define in particular a setting to couple diffeomorphic deformations with the action of finite dimensional Lie groups, such as isometries or scalings. Chapters 8 and 9 then explore the anisotropy of shapes, which we characterize by metrics on the ambient space. We define various group actions on these metrics and therefore transport the anisotropic features of shapes.

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# Contents

# Chapter 1

# Introduction et résumé des contributions (Français)

L'objet de cette thèse est d'étendre les modèles de grandes déformations en imagerie médicale via des structures sous-Riemanniennes sur des groupes de dimension infinie.

On commence par une brève introduction du domaine, en particulier de l'analyse des espaces de formes, et de la formalisation géométrique des différents problèmes de recalage difféomorphique.

## Analyse de formes en anatomie computationnelle

En quelques décennies, l'imagerie médicale a révolutionné le diagnostic et la prise en charge des pathologies dans tous les secteurs de la santé. Dans ces diverses modalités, elle a par exemple permis d'accéder à une information plus précise, notamment des tissus en 3D ou de données de transcriptomique spatiale. De par l'émergence de ces nouvelles techniques et technologies d'acquisition de données toujours plus performantes, l'analyse de formes a ainsi connu un large développement ces dernières années. En effet l'enjeu est d'obtenir une interprétation automatique de ces données d'imagerie, qui peuvent se présenter sous forme d'images (via des IRM par exemple), de courbes, surfaces, de nuages de points, etc.

L'idée est que la complexité apparente de ces données, leur nombre toujours croissant ne permet pas forcément une analyse totale visuelle par un observateur humain. Il s'agit alors de donner un cadre géométrique afin de permettre l'analyse statistique de ces images et données.

Pour formaliser ce problème, on définit tout d'abord un espace de formes $\mathcal{Q}$ sur lequel on pourra travailler. Il s'agit en général d'une variété différentielle, souvent de dimension infinie permettant d'encoder toute la complexité géométrique des données. L'idée est alors de définir sur cet espace des métriques ou des problèmes variationnels permettant de comparer les formes, et de développer des outils statistiques dessus. Une première approche ayant donné lieu à une riche littérature consiste à définir des structures Riemanniennes *intrinsèques* sur ces espaces [68, 69, 15, 14].

Dans cette thèse, on se placera plutôt sur une approche initiée par d'Arcy Thompson [92] qui consiste à comprendre les différences entre les formes via des déformations ambiantes les transformant. Cette idée a donné lieu à de nombreuses méthodes étudiant les espaces de formes à travers des actions de groupes de déformations [23, 30, 95, 93]. En particulier les travaux [23, 30] introduisent des déformations engendrées par un groupe de difféomorphismes. Ces difféomorphismes sont en fait générés par des champs de vecteurs $v_t$ en résolvant l'équation différentielle

$$\dot{\varphi}_t = v_t \circ \varphi_t. \tag{1.1}$$

Il s'agit alors simplement de ne pas manipuler directement ces déformations, mais plutôt les champs de vecteurs modélisant des déplacements infinitésimaux. Les problèmes variationnels sur l'espace de formes $\mathcal{Q}$ sont alors déduits de problèmes variationnels sur les champs de vecteurs engendrant les déformations. En particulier, on présente le formalisme LDDMM (Large deformation diffeomorphic metric mapping) introduit par exemple dans [95, 93, 17, 22].

# Le formalisme LDDMM

## Le problème d'appariement d'images

Voyons, sans rentrer dans les détails techniques, comment on peut obtenir le cadre de LDDMM, qui sera le cadre principal de cette thèse à partir d'un exemple d'appariement d'images. Le problème est le suivant : étant données deux images, une source $I_S : \mathbb{R}^d \to \mathbb{R}^d$ et une cible $I_T : \mathbb{R}^d \to \mathbb{R}^d$, l'objectif est de trouver un difféomorphisme $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ envoyant la source sur la cible :

$$I_S \circ \varphi^{-1} = I_T \tag{1.2}$$

On voit alors déjà ici apparaître une action d'un groupe de déformations $\mathrm{Defo}(\mathbb{R}^d) = \{\varphi : \mathbb{R}^d \to \mathbb{R}^d \mid \varphi \text{ inversible}\}$ sur l'espace des images $\mathcal{I}$. Bien évidemment, ce problème posé ainsi est très difficile, et il n'est pas forcément souhaitable de matcher exactement $I_S$ et $I_T$ en particulier en présence de bruit. Une première relaxation est alors de considérer un terme de dissimilarité sur l'espace des images $\mathcal{D} : \mathcal{I}^2 \to \mathbb{R}$ et de minimiser

$$\inf_{\varphi} \mathcal{D}(I_S \circ \varphi^{-1}, I_T). \tag{1.3}$$

Ce nouveau problème d'appariement glouton et déjà bien étudié [29] est encore limité, notamment car les déformations obtenues manquent de régularité et peuvent donner lieu à des transformations non réalistes. Pour gérer ce problème on peut alors rajouter un coût sur le groupe des déformations et étudier

$$\inf_{\varphi} d_{\mathrm{Diff}}(\mathrm{id}, \varphi) + \mathcal{D}(I_S \circ \varphi^{-1}, I_T) \tag{1.4}$$

avec $d_{\mathrm{Diff}}$ une distance sur l'espace des difféomorphismes. En reprenant l'idée fondamentale [95, 93] que l'on peut construire ces difféomorphismes en intégrant des champs de vecteurs via l'équation (1.1) on peut définir le nouveau problème d'appariement :

$$\inf_{v_t} \frac{1}{2} \int_0^1 |v_t|_V^2 dt + \mathcal{D}(I_S \circ (\varphi^{v_t})^{-1}, I_T) \tag{1.5}$$

où $|\cdot|_V$ est une norme pré-Hilbertienne sur un espace de champs de vecteurs $V \subset \Gamma(\mathbb{R}^d)$. Plusieurs choix pour l'espace $V$ et son produit scalaire sont alors possibles, engendrant une structure invariante à droite sur le groupe des difféomorphismes $\mathrm{Diff}(\mathbb{R}^d)$.

Par exemple, les structures Riemanniennes invariantes à droite sur le groupe des difféomorphismes ont été particulièrement étudiées dans la littérature, notamment de par les connexions avec la dynamique des fluides et la mécanique géométrique [10, 32, 49, 20, 75, 56]. Dans ces approches, on considère des groupes de difféomorphismes lisses avec certaines contraintes de régularité au bord (par exemple à support compact, ou de régularité Sobolev), et les métriques Riemanniennes sont alors *faibles* (dans le sens où la topologie induite par la métrique est plus faible que la topologie originale du groupe de difféomorphismes, voir par exemple [14]).

Une autre approche donnant lieu aussi à des structures invariantes à droite serait de considérer un sous-espace de champs de vecteurs générés par un noyau reproduisant [40, 39]. L'intérêt est qu'un tel noyau permet d'effectuer très rapidement de nombreux calculs liés à la minimisation du problème (1.5). La structure obtenue ainsi n'est cependant pas Riemannienne mais sous-Riemannienne telle que décrite dans [8, 5]. Les auteurs présentent alors dans ces travaux une généralisation de la géométrie sous-Riemannienne (cf. [73]) et ses outils à la dimension infinie et en particulier au groupe des difféomorphismes, permettant une description géométrique des méthodes LDDMM. Cette thèse se place dans la continuité de ce cadre. On résume dans la section suivante les principaux résultats obtenus pour ces structures invariantes à droite sur les groupes des difféomorphismes.

## Formalisation géométrique par les structures sous-Riemanniennes

Soit $M$ une variété différentielle de dimension finie, à géométrie bornée (par exemple $M = \mathbb{R}^d$). Soit $s \geq d/2+1$, notons $H^s(M, M)$ l'espace des applications de régularité $H^s$ de $M$ (cf. par exemple [32, 33, 80]) et définissons suivant [8] le groupe des difféomorphismes de régularité Sobolev :

$$\mathrm{Diff}_{H^s}(M) = \mathrm{Diff}_{C^1}(M) \cap H^s(M, M)$$

Il est déjà bien connu que $\mathrm{Diff}_{H^s}(M)$ est une variété de Hilbert et un groupe topologique [53], et on a de plus

$$T_\varphi \, \mathrm{Diff}_{H^s}(M) = \Gamma_{H^s}(TM) \circ \varphi, \quad \varphi \in \mathrm{Diff}_{H^s}(M)$$

où $\Gamma_{H^s}(TM)$ est l'ensemble des champs de vecteurs sur $M$ de régularité $H^s$.

Dans [8], les auteurs construisent alors des structures sous-Riemanniennes *fortes* invariantes à droite comme des triplets $(\mathrm{Diff}_{H^s}(M), V, \langle \cdot, \cdot \rangle_V)$, avec $(V, \langle \cdot, \cdot \rangle_V)$ un espace de Hilbert (par exemple un RKHS) inclus continûment dans $\Gamma_{H^s}(TM)$. Cela permet de définir alors un sous-fibré $\mathcal{H} \subset T\, \mathrm{Diff}_{H^s}(M)$ du fibré tangent par composition à droite

$$\mathcal{H}_\varphi = V \circ \varphi \subset T_\varphi \, \mathrm{Diff}_{H^s}(M)$$

ainsi qu'une métrique Riemannienne forte sur ce sous-fibré

$$\langle v \circ \varphi, v \circ \varphi \rangle_\varphi = \langle v, v \rangle_V, \quad \forall \varphi \in \mathrm{Diff}_{H^s}(M), \; v \circ \varphi \in \mathcal{H}_\varphi$$

La difficulté principale, par rapport à la géométrie Riemannienne plus classique, est que l'on doit maintenant se restreindre à des courbes dans $\mathrm{Diff}_{H^s}(M)$ dont la dérivée temporelle appartient à un sous-fibré du fibré tangent. Se posent alors les questions usuelles de contrôlabilité (exacte ou approchée), ainsi que des différents résultats de complétude (métrique, géodésique, existence de solutions à l'équation des géodésiques) qui sont démontrés dans [8].

Dès lors, en supposant maintenant que ce groupe de difféomorphismes agit sur l'espace des formes $\mathcal{Q}$, sous de bonnes conditions de régularité [6, 9], on peut de même induire une structure sous-Riemannienne sur $\mathcal{Q}$ et établir une formulation géométrique du problème de minimisation (1.5). En particulier, ces approches permettent des résultats de caractérisation des minimiseurs et points critiques de (1.5).

### Vers une extension de LDDMM

Bien que le formalisme présenté précédemment permette d'obtenir des déformations riches et ainsi résoudre de nombreux problèmes de recalage, quelques extensions ont été développées afin d'obtenir des problèmes plus complexes et des appariements plus précis. En particulier, le formalisme des métamorphoses [94, 50] pour l'appariement d'images rajoute un terme d'ajout d'intensité dans la déformation, permettant un recalage entre images ayant des différences de topologie [35, 51]. Le groupe de déformations alors considéré est maintenant un produit semi-direct du groupe des difféomorphismes avec un espace fonctionnel modélisant les ajouts d'intensité, sur lequel on définit aussi une métrique sous-Riemannienne.

Plus récemment des approches multi-échelles ont aussi été développées afin d'obtenir des recalages avec plusieurs degrés de précision simultanément [18, 84, 89, 71]. Ces méthodes considèrent un espace à noyau reproduisant généré par une somme de noyaux, chaque noyau représentant une échelle différente. On peut alors réécrire le problème de recalage en utilisant l'action d'un produit de groupes de difféomorphismes sur l'espace des formes, chaque difféomorphisme étant associé à un noyau et agissant à une échelle précise. Poursuivant cette idée, dans [71] les auteurs considèrent que l'espace de formes peut aussi s'écrire comme un produit de variétés, permettant d'écrire une forme sous ces différentes échelles. Les auteurs introduisent alors une stratégie de recalage du grossier au fin "coarse-to-fine", avec des interactions entre les échelles.

Ces différents nouveaux problèmes indiquent alors bien que le cadre LDDMM de recalage via l'action d'un seul groupe de difféomorphismes est limité, et on se propose dans cette thèse de généraliser et étendre cette approche à des groupes de dimension infinie en général.

## Les groupes de Lie en dimension infinie

Rentrons un peu plus dans le vif du sujet et évoquons le sujet des groupes de Lie en dimension infinie. Comme nous l'avons évoqué précédemment, c'est un sujet riche et ayant donné lieu à une vaste littérature notamment de par son application à la physique et en particulier à la dynamique des fluides [10, 32]. Ceci a notamment permis de révéler des phénomènes n'ayant lieu qu'en dimension infinie.

En particulier, il est souvent inévitable de sortir du cadre des espaces de Banach quand on travaille avec les groupes de Lie en dimension infinie, notamment avec le théorème d'Omori :

**Theorem 1.1** ([78])**.** *Soit $G$ un groupe de Lie-Banach connexe, et supposons que $G$ agit transitivement, fidèlement et de manière lisse sur une variété compacte. Alors $G$ est un groupe de Lie de dimension finie.*

Or on a bien vu que le formalisme de grandes déformations formulé précédemment pour l'étude des espaces de formes se base sur une action d'un groupe sur une variété. Cela implique, si on veut travailler avec des groupes de Lie de considérer des espaces fonctionnels de Fréchet (ou au-delà), ce qui peut poser des difficultés analytiques car on ne dispose plus forcément des théorèmes classiques de Cauchy-Lipschitz, d'inversion locale, etc. (cf. par exemple [79] pour une étude de certains types de groupes de Lie en dimension infinie).

Une autre possibilité, qui sera adoptée dans cette thèse, est d'accepter de rester dans un cadre de Banach mais de perdre la différentiabilité de la composition dans le groupe ou de l'application inverse [63, 16]. En particulier, l'exemple du groupe des difféomorphismes de régularité de Sobolev $H^s$ n'est pas un groupe de Lie (et en général, les groupes de difféomorphismes avec une régularité finie ne sont de même pas des groupes de Lie).

# Résumé des contributions

L'objet de cette thèse est donc d'étendre les résultats de géométrie sous-Riemannienne invariante à droite sur les groupes des difféomorphismes [8], à des groupes, et plus précisément des *half-Lie* groupes en dimension infinie. En particulier, on aimerait retrouver les principaux résultats de complétude et pouvoir caractériser les géodésiques normales (cf. [73]) via des équations Hamiltoniennes. De plus, on aimerait utiliser ces structures et les induire sur les espaces de formes classiques (courbes, surfaces, images, landmarks) via des actions de ces groupes, étendant alors le cadre défini en [5, 9, 6]. On peut ainsi explorer de nouvelles applications et de nouveaux problèmes de recalage, on pense notamment aux méthodes multi-échelles.

Cette thèse se concentre principalement sur les constructions théoriques et la compréhension géométrique des espaces de formes, et des actions que l'on peut y définir dessus. Quelques exemples numériques, simples et minimalistes, sont proposés, afin de mettre à l'épreuve ces modèles. Cependant ce travail demeure presque exclusivement dédié au développement d'outils et résultats théoriques. En particulier, des applications numériques, notamment en imagerie médicale, nécessiteraient des approfondissements supplémentaires, ainsi qu'un travail de modélisation et d'analyse plus poussé.

Ce manuscrit se découpe alors en trois parties, en allant du plus abstrait au plus appliqué. Les chapitres 3 et 4 présentent de manière générale le calcul différentiel et intégral sur les espaces et variétés de Banach, introduisant les principaux outils nécessaires avant d'étudier les structures sous-Riemanniennes. En particulier on y introduit la notion d'half-Lie groups, qui seront les objets principaux dans ce manuscrit. Ensuite les chapitres 5 et 6 introduisent et étudient les structures sous-Riemanniennes fortes invariantes à droite sur ces groupes ainsi que celles induites sur les espaces de formes. Enfin dans les chapitres 7, 8 et 9, on étudie quelques applications au recalage multi-échelle et à des transports anisotropiques. C'est en particulier dans ces chapitres que l'on introduira les exemples usuels d'espaces de formes (courbes, landmarks, images) ainsi que des expériences numériques.

Nous présentons dans les prochaines sections les principales notions et les principaux résultats de cette thèse.

## Courbes absolument continues

Afin d'étudier des structures sous-Riemanniennes et les différents problèmes de contrôle optimal, il est nécessaire d'avoir un bon cadre pour étudier des équations différentielles ordinaires dans les variétés de Banach et d'introduire les bons espaces de courbes. On présente ici un résultat du chapitre 3 (section 3.2).

On considère ici les équations différentielles au sens de Carathéodory [27, 98], et on intègre alors sur des variétés des équations avec une régularité en temps $L^p$, avec $p \geq 1$. Pour cela, on se place dans le cadre des courbes absolument continues tel que défini par Glockner [41], que l'on résume rapidement. Soit $M$ une variété lisse modelée sur un espace de Banach $\mathbb{B}$, et $I \subset \mathbb{R}$ un intervalle compact de $\mathbb{R}$. L'espace des courbes $L^p$ absolument continues $AC_{L^p}(I, M)$ est défini comme l'ensemble des courbes continues $\eta : I \to M$ telles que pour toute carte locale $(U, \varphi)$ de $M$, et tout $a < b \in I$ avec $\eta([a, b]) \subset U$, alors la courbe $\varphi \circ \eta : [a, b] \to \mathbb{B}$ est absolument continue dans $\mathbb{B}$, i.e.

$$\forall t \in [a, b],\ \varphi \circ \eta(t) = \varphi \circ \eta(a) + \int_a^t \gamma(s) ds$$

avec $\gamma \in L^p([a, b], \mathbb{B})$. On peut alors construire une structure différentielle sur l'espace $AC_{L^p}(I, M)$, ce qui est essentiel pour ensuite étudier les structures sous-Riemanniennes plus tard:

**Theorem 1.2.** *On obtient la structure différentielle suivante sur $AC_{L^p}(I, M)$*

1. *L'espace $AC_{L^p}(I, M)$ est une variété de Banach*
2. *Pour $t \in I$, l'évaluation*

$$\mathrm{ev}_t : \begin{cases} AC_{L^p}(I, M) & \longrightarrow & M \\ \eta & \longmapsto & \eta(t) \end{cases}$$

   *est lisse.*

3. *Le fibré vectoriel $AC_{L^p}(I, TM) \to AC_{L^p}(I, M)$ définit le fibré tangent. Pour $\eta \in AC_{L^p}(I, M)$, on a alors*

$$T_\eta AC_{L^p}(I, M) = AC_{L^p}(I \leftarrow \eta^* M)$$

   *où $AC_{L^p}(I \leftarrow \eta^* M) = \{\gamma \in AC_{L^p}(I, TM),\ \gamma(t) \in T_{\eta(t)} M,\ \forall t \in I\}$*

**Remark 1.3.** *Un récent travail de Pinaud [83] propose aussi une structure différentielle pour l'espace $AC_{L^p}(I, M)$, en supposant l'existence d'une addition locale, et en généralisant aux variétés $M$ modelées sur des espaces localement convexes. La structure différentielle obtenue est alors équivalente à la structure développée dans cette thèse.*

## Half-Lie groups (groupes de type demi-Lie)

On présente ici les résultats principaux du chapitre 4, que l'on peut retrouver notamment dans [82, 81].

On introduit la notion d'half-Lie groups modelés sur des espaces de Banach tels que définis dans [63, 16] :

**Definition 1.4.** Un half-Lie group de Banach est un groupe topologique muni d'une structure différentielle lisse Banachique, telle que la multiplication à droite $R_{g'} : G \to G, g \mapsto gg'$ est lisse pour tout $g' \in G$.

**Remark 1.5.** *L'auteur de la thèse ne connaît pas le vocabulaire français désignant les "half-Lie groups". On propose ici, dans cette introduction en français, la terminologie de "groupe de type demi-Lie" (et non demi-groupe de Lie qui pourrait entrainer des confusions avec la théorie des semi-groupes.)*

On note alors que sur ces groupes la multiplication à gauche est seulement continue, et de nombreux outils usuels de la théorie des groupes de Lie (l'application exponentielle, le crochet de Lie, etc.) ne peuvent être importés directement. Pour remédier à ce manque de régularité, on étudie, de manière analogue à [79, 16] des familles d'half-Lie groups $\{G^k, k \geq 1\}$ vérifiant certaines propriétés de régularité :

**(G.1)** $G^{k+1}$ est un sous-groupe de $G^k$ avec inclusion lisse

**(G.2)** Pour tout $l \geq 0$, l'application inverse $\mathrm{inv} : G^{k+l} \to G^k$ est $C^l$.

**(G.3)** Pour tout $l \geq 0$, la multiplication

$$\begin{array}{ccc} G^{k+l} \times G^k & \longrightarrow & G^k \\ (g', g) & \longmapsto & g'g \end{array}$$

est $C^l$ et $C^\infty$ par rapport à la première variable $g'$, pour $g$ fixé.

**(G.4)** Pour tout $l \geq 0$, la multiplication infinitésimale à droite

$$\begin{array}{ccl} G^k \times T_eG^{k+l} & \longrightarrow & TG^k \\ (g, u) & \longmapsto & u \cdot g = \partial_{g'}(g'g)_{|g'=e}(u) = T_eR_g(u) \in T_gG^k \end{array}$$

est $C^l$, et $C^\infty$ par rapport à la première variable.

**(G.5)** La multiplication infinitésimale à gauche

$$\begin{array}{ccc} G^{k+1} \times T_eG^k & \longrightarrow & TG^k \\ (g, u) & \longmapsto & T_eL_g(u) \end{array}$$

est un morphisme de fibrés vectoriels $C^1$.

En particulier à partir de ces hypothèses, on peut définir des équivalents des représentations adjointes et du crochet de Lie

$$[\cdot, \cdot]_k : T_eG^{k+1} \times T_eG^{k+1} \to T_eG^k.$$

On observe d'ailleurs une perte de régularité pour ces crochets : l'espace $T_eG^{k+1}$ n'est pas stable par $[\cdot, \cdot]_k$.

### Régularité $L^p$

Une des première questions que l'on se pose ensuite concerne la régularité des groupes de type demi-Lie, à savoir la possibilité d'intégrer des courbes de l'espace tangent en l'identité en des courbes sur tout le groupe. C'est essentiel pour pouvoir faire le lien entre le point de vue Eulérien et Lagrangien dans les futurs problèmes variationnels, notamment lorsque l'on introduit des problèmes invariants à droite. En particulier, dans [16], les auteurs montrent sous certaines hypothèses la régularité $C^\infty$ de ces groupes, au sens où on peut intégrer des courbes lisses en temps de l'espace tangent en l'identité.

On généralise dans cette thèse ces résultats aux courbes $L^p$ en temps:

**Proposition 1.6.** *Soit* $\{G^k,\, k \geq 0\}$ *une famille de groupes de type demi-Lie vérifiant les propriétés (G.1-5),* $k \geq 0$*, et* $I \subset \mathbb{R}$ *un intervalle et* $t_0 \in I$*. Pour tout* $u \in L^p(I, T_eG^{k+1})$ *l'équation différentielle*

$$\begin{cases} \dot{g}(t) = u(t) \cdot g(t) = T_eR_{g(t)}(u(t)) \\ g_{t_0} = e \end{cases} \tag{1.6}$$

*admet une unique solution globale (i.e. définie sur* $I$*)* $g \in AC_{L^p}(I, G^k)$*.*
*De plus, si* $u \in L^p(I, T_eG^{k+n})$*, avec* $n \geq 1$*, alors pour tout* $t \in I$*, la translation à gauche* $L_{g(t)} : G^k \to G^k$ *est* $C^n$*.*

En particulier on peut alors définir l'application d'évolution

$$\mathrm{Evol} : L^p(I, T_eG^k) \to AC_{L^p}(I, G^k)$$

en résolvant pour tout $u \in L^p(I, T_eG^k)$, l'équation (1.6). Il est ensuite naturel d'étudier la régularité de cette application, et on démontre la proposition suivante

**Proposition 1.7.** *Soit* $I$ *un intervalle compact de* $\mathbb{R}$*.*

1. *La restriction de l'application d'évolution* $\mathrm{Evol}_{G^k} : L^p(I, T_eG^{k+1+l}) \to AC_{L^p}(I, G^k)$*, avec* $l \geq 0$ *est* $C^l$*.*

2. *Pour tout* $u, \delta u \in L^p(I, T_eG^{k+1+l})$*, la différentielle* $\delta g = T_u \mathrm{Evol}_{G^k}(\delta u) \in AC_{L^p}(I \leftarrow g^*TG^k)$ *est l'unique solution du problème de Cauchy linéaire suivant*

$$\delta\dot{g}(t) = \partial_g(T_eR_g u(t))_{|g=g(t)}\delta g(t) + \partial_u(T_eR_{g(t)}u)_{|u=u(t)}\delta u(t),\ \delta g(0) = 0 \tag{1.7}$$

*où* $g(t) = \mathrm{Evol}_{G^k}(u)(t)$*.*

### Structures de Poisson et réduction de Lie-Poisson

Les structures sous-Riemanniennes (que l'on introduira par la suite) sur les groupes de type demi-Lie font intervenir des formulations Hamiltoniennes pour la caractérisation des géodésiques. De plus si on suppose ces structures invariantes à droite, on obtient des Hamiltoniens $H : T^*G \to \mathbb{R}$ invariant à droite sur le cotangent, c'est à dire vérifiant

$$H \circ T^*R_g = H$$

Ces propriétés d'invariance permettent d'obtenir des réductions de la dynamique. On fait alors le lien avec la géométrie de Poisson, en dotant en particulier l'espace tangent à l'identité d'un équivalent du crochet de Lie-Poisson.

Ces phénomènes ont été beaucoup étudiés ([65] par exemple) pour des Hamiltoniens invariants à droite sur des groupes de Lie en dimension finie, donnant alors les théorèmes standards de réduction de Lie-Poisson et d'Euler-Poincaré (cf. aussi [64]). Dans le cas de la dimension infinie, les résultats nécessitent un peu plus de travail car il devient notamment plus difficile de définir des structures de Poisson, les espaces n'étant pas réflexifs entrainant des formes symplectiques et des crochets de Poisson au sens faible

[76, 77, 43]. On définit par exemple une structure de Poisson faible associée à la variété symplectique faible $(T^*G, \omega)$ comme un couple $(\mathcal{A}^\infty_{T^*G}, \{\cdot,\cdot\}_\omega)$ où $\mathcal{A}^\infty_{T^*G}$ est une sous-algèbre de $C^\infty(T^*G)$ définie par

$$\mathcal{A}^\infty_{T^*G} \coloneqq \{H \in C^\infty(T^*G) \mid H \text{ admet un gradient symplectique}\}.$$

et $\{\cdot,\cdot\}_\omega$ est le crochet de Poisson usuel associé à la forme symplectique $\omega$ :

$$\{H, F\} = \omega(X_F, X_H)$$

pour tout $H, F \in \mathcal{A}^\infty_{T^*G}$, avec $X_H$ et $X_F$ leurs gradients symplectiques.

Un autre problème ici est que l'espace tangent en l'identité n'est pas une algèbre de Lie comme on l'a vu : on perd un degré de régularité sur les groupes de type demi-Lie quand on applique le crochet de Lie $[\cdot,\cdot]_k$. On ne peut donc pas définir comme habituellement le crochet de Lie-Poisson sur le dual de $T^*_eG$. On peut cependant, comme pour le crochet de Lie, autoriser des pertes de régularité et obtenir des algèbres de Poisson non stables pour le crochet. On définit ainsi

$$\mathcal{A}^l_{T^*G} \coloneqq \{H \in C^l(T^*G) \mid H \text{ admet un gradient symplectique}\}$$

et

$$\mathcal{A}_{T^*_eG^l} \coloneqq \{h \in C^\infty(T^*_eG^l) \mid dh \in C^\infty(T^*_eG^l, T_eG^l)\}$$

ainsi que le crochet de Poisson sur $\mathcal{A}_{T^*_eG^{l+1}}$

$$\forall m \in T^*_eG^l,\ \{h, f\}_+(m) \coloneqq (m \mid [dh(m), df(m)]_l)$$

permettant d'obtenir un élément de $\mathcal{A}_{T^*_eG^l}$. L'introduction de l'application moment $m^l = T^*G \to T^*_eG^l : (g,p) \mapsto T_eR^*_g p$ nous permet de retrouver un analogue de la réduction de Lie-Poisson:

**Proposition 1.8.** *On a les points suivants :*

1. $\{\mathcal{A}^{l+1}_{T^*G}, \mathcal{A}^{l+1}_{T^*G}\}_\omega \subset \mathcal{A}^l_{T^*G}$ *et* $\{\mathcal{A}_{T^*_eG^{l+1}}, \mathcal{A}_{T^*_eG^{l+1}}\}_+ \subset \mathcal{A}_{T^*_eG^l}$.
2. *L'application* $m^l$ *est* $C^{l-1}$ *et* $(m^l)^*\mathcal{A}_{T^*_eG^l} \subset \mathcal{A}^{l-1}_{T^*G}$
3. *On a*
$$\forall h, f \in \mathcal{A}_{T^*_eG^l},\ \{h, f\}_+ \circ m^l = \{h \circ m^l, f \circ m^l\}_\omega. \tag{1.8}$$

De plus, dans le cadre des Hamiltoniens invariants à droite, ce premier résultat permet de faire le lien entre la dynamique Hamiltonienne dans le cotangent du groupe $T^*G$ et la dynamique associée au crochet de Poisson dans $T^*_eG^l$

**Proposition 1.9.** *Soit* $f \in \mathcal{A}_{T^*_eG^{l+1}}$*, et définissons le champs de vecteurs lisse* $\hat{X}_f \in C^\infty(T^*_eG^l, T^*_eG^{l+1})$ *par*

$$\forall m \in T^*_eG^l,\ \hat{X}_f(m) = -\operatorname{ad}^*_{df \circ i^*_{l+1}(m)} m.$$

*Alors, pour tout* $h \in \mathcal{A}_{T^*_eG^{l+1}}$*, on a*

$$\forall m \in T^*_eG^l,\ \{h, f\}_+(m) = \mathcal{L}_{\hat{X}_f} h(m)$$

**Remark 1.10.** *Ces résultats sont aussi des analogues pour la géométrie Hamiltonienne des résultats "no loss, no gain" d'Ebin et Marsden [32] sur les sprays invariants à droite, venant en particulier de structures Riemanniennes invariantes à droite.*

## Géométrie sous-Riemannienne invariante à droite forte et applications aux espaces de formes

On peut étudier ensuite avec l'introduction de tous ces outils les géométries sous-Riemanniennes fortes et invariantes à droite sur les groupes de type demi-Lie. On résume ici les résultats principaux démontrés en chapitres 5 et 6, que l'on peut retrouver en particulier dans [81, 82].

### Structures sous-Riemanniennes sur les groupes de type demi-Lie

Soit $\{G^k, k \geq 0\}$ une famille de groupes de type demi-Lie vérifiant (G.1-5). Suivant le formalisme de [7, 8], on introduit alors, pour $V \subset T_eG$ un espace de Hilbert, la structure sous-Riemannienne associée comme le triplet $(G \times V, T_eR, \langle\cdot,\cdot\rangle_V)$ où $\langle\cdot,\cdot\rangle_V$ est le produit scalaire sur $V$ et $T_eR$ la dérivée de la multiplication à droite donnant un morphisme de fibrés vectoriels

$$T_eR_g : G \times V \to TG.$$

On peut notamment rappeler le vocabulaire associé à ces structures. On considère les courbes horizontales $g : I \to G$ qui sont les courbes absolument continues dans $G$ telles qu'il existe un lift $u(t) \in L^p(I, V)$ tel que

$$\dot{g}(t) = T_eR_{g(t)}u(t).$$

On peut ensuite définir, en utilisant la norme sur $V$, la longueur et l'énergie pour un tel système $(g, u)$:

$$L(g,u) = \int_I |u(t)|_V dt \qquad \text{et} \qquad E(g,u) = \frac{1}{2}\int_I |u(t)|_V^2 dt.$$

Ainsi, on munit $G$ d'un structure métrique pour la distance sous-Riemannienne suivante

$$d_V(g_0, g_1) = \inf\{L(g,u) \mid (g,u) \text{ est un système horizontal } L^1 \text{ d'extrémités } g_0 \text{ et } g_1\}.$$

Les principales propriétés de cette distance sont résumées dans la proposition suivante :

**Theorem 1.11.** *La distance sous-Riemannienne $d_V$ est une vraie distance, invariante à droite, et la topologie induite par cette distance est plus fine que la topologie intrinsèque sur $G$. De plus l'espace $(G, d_V)$ est complet, et s'il existe une topologie Hausdorff sur $G$ telle que l'application finale* $\mathrm{End} : L^2(I, V) \to G, u \mapsto \mathrm{Evol}(u)(1)$ *soit faiblement continue, alors toute paire de points $g_0, g_1 \in G$ à distance finie peut être connectée par une géodésique minimisante.*

L'étape suivante est alors de caractériser les points critiques de l'énergie, afin de pouvoir calculer la distance sous-Riemannienne, et à terme résoudre en particulier le problème (1.5). En dimension infinie, il existe trois types de géodésiques [7], notamment à cause de l'absence de surjectivité de l'application point final End (voir par exemple [73]).

On se concentrera ici seulement sur les géodésiques sous-Riemanniennes *normales* qui sont les courbes horizontales absolument continues $g_t$ telles qu'il existe des multiplicateurs de Lagrange $(\lambda, p) \in \mathbb{R} \times T^*_{g_1}G$, avec $\lambda \neq 0$ vérifiant

$$\lambda dE(u) + d\,\mathrm{End}(u)^* p = 0$$

On verra en particulier que ces géodésiques sont liées aux problèmes de recalage inexact que l'on introduira par la suite. Pour ces géodésiques, il existe alors une caractérisation Hamiltonienne. En effet, on définit l'Hamiltonien associé à la structure sous-Riemannienne

$$H(g, p) = \frac{1}{2}|K_V T_e R_g^* p|_V^2,$$

où $K_V : V^* \to V$ est l'inverse de l'isométrie de Riesz, et on remarque ici les courbes horizontales vérifiant (1) sont exactement les flots de ce Hamiltonien. On obtient ensuite un résultat d'existence globale du flot Hamiltonien sous certaines conditions:

**Proposition 1.12.** *On suppose que l'espace de Hilbert $V$ est continûment inclus dans $T_eG^2$. Alors, pour tout $p \in T_e^*G$ covecteur initial, il existe une unique courbe globale $(g_t, p_t) \in AC_{L^2}(I, T^*G)$ satisfaisant les équations géodésiques normales:*

$$(\dot{g}_t, \dot{p}_t) = \nabla^\omega H(g_t, p_t) \quad (g_0, p_0) = (e_G, p) \tag{1.9}$$

Notons de plus que le Hamiltonien $H$ défini est invariant pour l'action à droite sur le cotangent $T^*G$, et en particulier la discussion précédente sur la réduction de Poisson s'applique. On obtient alors une nouvelle caractérisation de ces géodésiques par une nouvelle équation sur le dual $T_e^*G$. En effet, on peut définir pour toute courbe $(g_t, p_t) \in AC_{L^2}(I, T^*G)$ le moment associé

$$m_t = T_e R_{g_t}^* p_t.$$

Dans le cas de géodésiques normales, ce moment satisfait alors l'équation d'Euler-Arnold-Poincaré (sous-Riemannienne) :

**Theorem 1.13.** *On suppose que $V \hookrightarrow T_eG^2$. Soit $g \in \mathrm{AC}_{L^2}(I, G)$ une courbe horizontale, avec $u_t = \dot{g}_t g_t^{-1} = (T_e R_{g_t})^{-1}\dot{g}_t$ la vitesse Eulerienne. Alors, $g_t$ est une géodésique normale si et seulement s'il existe un moment $m_t \in C(I, T_e^*G^1) \cap C^1(I, T_e^*G^2)$, avec $K^1 m_t = u_t$ et qui satisfait l'équation d'Euler-Arnold-Poincaré sous-Riemannienne (dans $T_e^*G^1$) :*

$$\dot{m}_t + \mathrm{ad}_{u_t}^* m_t = 0. \tag{1.10}$$

*Dans ce cas, le covecteur $p(t) = (T_e R_{g_t}^*)^{-1} m(t)$ définit une courbe dans le cotangent $(g, p) \in \mathrm{AC}_{L^2}(I, T^*G)$ solution du flot Hamiltonien normal* (1.9).

L'équation (1.10) est notamment bien connue dans le cadre de la géométrie sous-Riemannienne sur les groupes des difféomorphismes [8, 50] sous le nom d'EPdiff, et son équivalent dans l'algèbre de Lie $T_eG$ a été tout particulièrement étudié (d'abord dans [32]) notamment en raison de son lien avec les métriques Riemannienne invariantes à droite et les applications à la dynamique des fluides [32, 65, 20, 75].

### Espaces de formes

On peut maintenant revenir à l'analyse des espaces de formes et utiliser ces métriques sous-Riemanniennes afin de définir les différents problèmes de recalage. On considère, étendant le cadre de [6], un espace de formes $\mathcal{Q}$, vu comme une variété différentielle de Banach sur laquelle agit une famille de groupes de type demi-Lie $\{G^k,\ k \geq 0\}$ vérifiant (G.1-5), en supposant en plus les conditions de régularité suivantes :

**(S.1)** *Continuité de l'action* : L'action $A : (g, q) \mapsto g \cdot q$ est continue

**(S.2)** *Action infinitésimale* : Pour tout $q \in \mathcal{Q}$, l'application $A_q = A(\cdot, q) : g \mapsto g \cdot q$ est $C^\infty$, et on note $\xi_q = \xi(\cdot, q) = \partial_g A(g, q)|_{g=e}$ sa différentielle en $e_G$.

**(S.3)** *Régularité de l'action* : Pour tout $l > 0$, les applications

$$A : \left\{ \begin{array}{ccc} G^k \times \mathcal{Q} & \longrightarrow & \mathcal{Q} \\ (g,\ q) & \longmapsto & g \cdot q \end{array} \right. \quad \text{et} \quad \xi : \left\{ \begin{array}{ccc} T_e G^k \times \mathcal{Q} & \longrightarrow & T\mathcal{Q} \\ (u,\ q) & \longmapsto & \xi_q(u) = u \cdot q \end{array} \right.$$

sont $C^k$.

Ces hypothèses sont des conditions assez naturelles pour induire une structure sous-Riemannienne sur $\mathcal{Q}$, et permet donc l'étude de nouveaux problèmes sur les espaces de formes avec des groupes de déformations diverses.

En effet, si $V \subset T_e G$ est un espace de Hilbert, on en déduit une structure sous-Riemannienne $(\mathcal{Q} \times V, \xi, \langle \cdot, \cdot \rangle_V)$. Les courbes horizontales $q \in AC_{L^p}(I, \mathcal{Q})$ pour cette structure sont alors les solutions de

$$\dot{q}_t = \xi_{q_t} u_t$$

où $u_t \in L^p(I, V)$ et se relèvent en des géodésiques horizontales sur le groupe : $q_t = g_t \cdot q_t$ avec $g_t = \mathrm{Evol}(u_t)$. On peut définir comme précédemment la longueur, l'énergie et la distance sous-Riemannienne $d_{\mathcal{Q}}$, de telle sorte que $(\mathcal{Q}, d_{\mathcal{Q}})$ soit un espace métrique complet. De plus les géodésiques normales pour cette structure seront alors le flot Hamiltonien donné par

$$H(q, p) = \frac{1}{2} |K_V \xi_q^* p|_V^2.$$

On voit ici apparaître de nouveau la variable moment $m_{\mathcal{Q}}(q, p) = \xi_q^* p \in T_e^* G$ et on retrouve notamment que cette variable vérifie l'équation d'Euler-Arnold-Poincaré. Cela permet de faire l'équivalence entre les géodésiques normales dans $\mathcal{Q}$ et celles dans $G$.

**Theorem 1.14.** *Soit $q \in AC_{L^2}(I, T\mathcal{Q})$ une courbe horizontale in $\mathcal{Q}$, et soit $g \in AC_{L^2}(I, TG)$ son relèvement minimal. Alors, $q_t$ est une géodésique normale si et seulement si $g_t$ est une géodésique normale dans $G$.*
*De plus, si $t \mapsto (g_t, p_t^G)$ et $t \mapsto (q_t, p_t^{\mathcal{Q}})$ sont solutions des équations normales, avec la condition initiale*

$$m_G(g_0, p_0^G) = m_{\mathcal{Q}}(q_0, p_0^{\mathcal{Q}}) \tag{1.11}$$

*alors les moments sont égaux en tout temps :*

$$\forall t \in I,\ m_{\mathcal{Q}}(q_t, p_t^{\mathcal{Q}}) = m_G(g_t, p_t^G).$$

### Problème inexact

On peut ensuite introduire une fonctionnelle $J : L^2(I, V) \to \mathbb{R}$ que l'on veut optimiser pour le recalage inexact, suivant l'exemple de 1.5, en ajoutant un terme d'attache aux données $\mathcal{D} : \mathcal{Q} \to \mathbb{R}$ :

$$J(u) = E(u) + \mathcal{D}(\mathrm{End}(u) \cdot q_S),$$

où $q_S$ est une forme source que l'on veut recaler à une cible. On démontre alors que les points critiques de cette fonctionnelle sont des géodésiques normales pour la structure sous-Riemannienne introduite.

## Applications aux espaces des formes

On propose dans les chapitres 7, 8 et 9 quelques applications aux espaces de formes de ces nouveaux outils. Le chapitre 7, dont la première partie provient d'un travail avec un autre étudiant de thèse Rayane Mouhli [74], propose des approches multi-échelle via actions de produits de groupes sur les espaces de formes. Le chapitre 8 continue cette idée en proposant en plus de rajouter aux formes une information d'anisotropie encodée par une matrice symétrique définie positive, que l'on transporte également. On construit alors des structures sous-Riemanniennes dépendantes de cette matrice, et on sort alors du cadre invariant à droite (bien que ces structures restent invariantes par l'action du groupe des difféomorphismes). Enfin le chapitre 9 provient d'un travail en cours, et propose un cadre d'exploration pour étudier des phénomènes d'anisotropie. On introduit alors le groupe des automorphismes du fibré tangent d'une variété compacte $M$, et son action naturelle par pousser en avant sur l'espace des métriques Riemanniennes de $M$. On développe alors le cadre différentiel et les propriétés de régularité de ces actions dans le but de pouvoir écrire par la suite des problèmes variationnels dessus.

### Recalage multi-échelle

On commence par étendre le cadre LDDMM et de la minimisation du problème 1.5 en rajoutant un terme venant de l'action d'un groupe de Lie $G$ de dimension finie :

$$\inf_{(X,v)\in L^2([0,1],\mathfrak{g}\times V)} J(X,v) = \int_0^1 \frac{1}{2}|X_t|_{\mathfrak{g}}^2 + \frac{1}{2}|v_t|_V^2 \, dt + \mathcal{D}(g_1, q_1) \quad (1.12)$$
$$\text{s.t} \quad \begin{cases} \dot{g}_t = X_t g_t \\ \dot{q}_t = X_t \cdot q_t + v_t \cdot q_t \\ (g_0, q_0) = (e_G, q_S) \end{cases}$$

avec $\mathfrak{g}$ l'algèbre de Lie de $G$, et $|\cdot|_{\mathfrak{g}}$ une norme euclidienne. On pense notamment pour $G$ à l'introduction des rotations, translations ou changements d'échelle dans la déformation afin d'offrir un recalage plus précis avec deux échelles. En particulier, on démontre, en adaptant une construction de [18], que l'on peut construire un half-Lie group adapté au problème (1.12) en définissant un produit semi-direct

$$G \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d).$$

On peut alors se placer dans le cadre de la géométrie sous-Riemannienne invariante à droite sur des half-Lie groups. L'enjeu est alors aussi de découpler les actions du groupe $G$ et du groupe des difféomorphismes, ce qui peut être possible en utilisant des propriétés d'invariance de la norme $|\cdot|_V$. On propose quelques applications pour un recalage combinant déformations rigides et non rigides.

On poursuit cette idée d'utiliser plusieurs groupes de déformations pour introduire un problème variationnel multi-échelles, suivant [71] :

$$\inf_{v^1,\ldots,v^L} J(v^1,..,v^l) = \frac{1}{2}\sum_{l=1}^{L}\int_I |v^l|^2_{V_l}\,dt + \mathcal{D}(\varphi_1 \cdot q_0^L) \tag{1.13}$$

$$\text{s.t} \quad \begin{cases} \varphi_0 = \text{id} \\ \dot{\varphi}_t = (\sum_{l=1}^{L} v^l) \circ \varphi_t \end{cases}$$

où les espaces $V_l, l \leq L$ sont des espaces de Hilbert tels que

$$V_0 \hookrightarrow V_1 \hookrightarrow \ldots \hookrightarrow V_L \hookrightarrow C_0^k(\mathbb{R}^d, \mathbb{R}^d)$$

permettant de modéliser chaque échelle. Ces espaces de Hilbert permettent de se ramener encore une fois à une structure sous-Riemannienne invariante à droite sur le groupe produit

$$\prod_{l\leq L} \text{Diff}_{C_0^k}(\mathbb{R}^d)$$

permettant de définir des schémas de minimisation "coarse-to-fine".

### Noyaux anisotropes

Dans le cadre du problème (1.12), on peut utiliser pour $V$ un espace à noyau reproduisant généré par un noyau Gaussien

$$k_\sigma(x,y) = \exp(-\frac{1}{2\sigma^2}|x-y|^2)$$

avec $\sigma > 0$. Dans certains problèmes de minimisation, on peut vouloir rajouter une information d'anisotropie et remplacer $\sigma$ par une matrice symétrique définie positive $\Sigma \in S_d^{++}$. En particulier, en appliquant une rotation ou un scaling aux formes, on est amené à transporter cette métrique $\Sigma$ pendant la dynamique afin que cette anisotropie suive la forme. On en revient à définir un nouveau problème variationnel (que l'on simplifie à une action des rotations et des difféomorphismes ici) :

$$\inf_{(A_t,u_t)\in L^2([0,1],\overline{V})} J(A_t,u_t) = \frac{1}{2}\int_0^1 |A_t|^2 + |u_t|^2_{V_{\Sigma_t}}\,dt + \mathcal{D}(R_1,\Sigma_1,q_1)$$

$$\text{s.t} \quad \begin{cases} \dot{q}_t = A_t \cdot q_t + u_t(q_t) \\ \dot{\Sigma}_t = A_t\Sigma_t - \Sigma_t A_t \\ \dot{R}_t = A_t R_t \end{cases} \tag{1.14}$$

On perd ici l'invariance à droite pour la structure sous-Riemannienne qui dépend maintenant de la métrique $\Sigma_t$ non constante.

### Anisotropie et transport de métriques

Enfin, le chapitre 9 correspond à un travail en cours, et propose un cadre différentiel pour explorer le transport de métriques sur une variété, permettant une nouvelle fois de rajouter des informations d'anisotropie. Supposons que l'espace ambiant est une variété compacte $M$ munie d'une métrique Riemannienne $g^M$. On introduit alors l'espace des métriques Riemanniennes $\mathrm{Met}_{C^k}(M)$, permettant d'encoder les informations d'anisotropie via des métriques sur $M$ que l'on compare à la métrique ambiante $g^M$. Le groupe des automorphismes du fibré tangent $\mathrm{Aut}_{C^k}(TM)$ agit naturellement sur l'espace des métriques via pousser en avant. On démontre en particulier que $\mathrm{Aut}_{C^k}(TM)$ est un Banach half-Lie group, ainsi que quelques propriétés de régularité de l'action.

Afin de seulement transporter les caractéristiques anisotropes, on se restreint en fait au sous-groupe des automorphismes préservant la métrique $g^M$

$$\mathrm{Aut}_{C^k,g^M}(TM) = \{\Psi \in \mathrm{Aut}_{C^k}(TM) \mid \Psi_* g^M = g^M\}$$

On établit ainsi un cadre différentiel afin d'explorer les questions de transport des métriques et d'anisotropie, dans le contexte de l'analyse des formes. On introduit par exemple un problème variationnel d'appariemment de métrique basé sur les modèles de grandes déformations

$$\begin{aligned} &\inf_{(u)\in L^2([0,1],\Gamma_{C^k}(TM))} J(u) = \frac{1}{2}\int_0^1 c(g_t,u_t)\,dt + \mathcal{D}(g_1) \\ &\text{tel que} \quad \begin{cases} \partial_t g(t) = -\nabla_{u(t)} g(t) - A_{u(t)} g(t) \\ g(0) = g_S \end{cases} \end{aligned} \tag{1.15}$$

avec $g_S$ une métrique source, $A_u g$ la partie anti-symmétrique de $\nabla_u$ appliquée à $g$, et $c(g,u) = \int_M \mathrm{tr}(g^{-1}(S_u g)g^{-1}(S_u g))\,\mathrm{Vol}(g)$ la métrique d'Ebin [31]. On propose quelques exemples numériques pour ce problème variationnel.

# Chapter 2

# Introduction

## Contents



The aim of this thesis is to extend large deformation models in medical imaging, by using sub-Riemannian structures on general infinite-dimensional groups. We begin with a brief overview of the field, with particular attention to shape analysis and the geometric formulation of diffeomorphic registration problems.

## 2.1 Shape analysis in computational anatomy

Over the past few decades, medical imaging has revolutionized the diagnosis of diseases. Through its diverse modalities, it has enabled precise access to information, for example of 3D tissues or through the acquisition of spatial transcriptomic data. The emergence of these new and more sophisticated data acquisition technologies has in turn driven significant progress in shape analysis. Indeed, the automatic interpretation of this imaging data and the development of new methods has become a major challenge.

Yet, the apparent complexity and ever-growing quantity of such data, which may take the form of images, surfaces, point clouds, etc., makes it pretty difficult for a human observer to perform a purely visual analysis. Therefore one of the goals of shape analysis is to establish a mathematical framework that enables the statistical analysis of these images and data.

To formalize this problem, we first define a shape space $\mathcal{Q}$ on which computations will be carried out. This space is typically a differentiable manifold, and often infinite dimensional, allowing to encode the full geometric complexity of the data. The idea is then to define on this space suitable metrics or variational problems to compare shapes.

A first approach consists in defining intrinsic Riemannian structures on these spaces and has led to a rich literature [68, 69, 15, 14].

In this thesis, however, we adopt another approach, originating from d'Arcy Thompson [92], which tries to understand differences between shapes through ambient deformations transforming them. This idea has given rise to numerous approaches studying shape spaces through the action of deformation groups [23, 30, 95, 93]. In particular, works as [23, 30] introduce deformations generated by diffeomorphisms. These diffeomorphisms are in fact generated by time-dependent vector fields through the ordinary differential equation

$$\dot{\varphi}_t = v_t \circ \varphi_t. \tag{2.1}$$

Therefore, rather than directly manipulating these deformations, we instead work with the vector fields, that can be thought as the infinitesimal displacements. Variational problems on the shape space $\mathcal{Q}$ can then be derived from variational problems defined on the space of vector fields generating the deformations. In particular, the LDDMM (Large Deformation Diffeomorphic Metric Mapping) framework, as introduced in [95, 93, 17, 22], is an example of such method. We present this framework in the next section.

## 2.2 LDDMM framework

### 2.2.1 The problem of image matching

Let's present heuristically a bit more this LDDMM setting, and we introduce an example of image matching. Given a source image $I_S : \mathbb{R}^d \to \mathbb{R}^D$, and a target image $I_t : \mathbb{R}^d \to \mathbb{R}^d$, the goal is to find the best deformation $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ deforming the source onto the target :

$$I_S \circ \varphi^{-1} = I_T \tag{2.2}$$

The left-hand side of this equation can be seen here as the action of the deformation $\varphi$ on the image $I_S$. Obviously, this first problem is very hard, and it is actually not necessary neither desirable to achieve an exact matching between the two images, particularly in the presence of noise. We therefore want to relax by introducing a discrepancy term $\mathcal{D} : \mathcal{I}^2 \to \mathbb{R}$ on the space of images $\mathcal{I}$. This leads to the minimization problem

$$\inf_{\varphi} \mathcal{D}(I_S \circ \varphi^{-1}, I_T) \tag{2.3}$$

This problem has been refered in the the literature as "greedy matching" [23]. However, the results produced are still not convincing, because the deformations obtained after the optimization are highly non regular and lead to non realistic matching. To tackle this new issue, we can then add a cost on the group of deformations, that penalizes non regular deformations:

$$\inf_{\varphi} d_{\mathrm{Diff}}(\mathrm{id}, \varphi) + \mathcal{D}(I_S \circ \varphi^{-1}, I_T) \tag{2.4}$$

where $d_{\mathrm{Diff}}$ is a distance on the space of diffeomorphisms of $\mathbb{R}^d$. Using the idea [95, 93] that diffeormophisms can be generated by integrating vector fields of $\mathbb{R}^d$ with equation (2.1), we can simplify the previous problem and define a new image matching problem:

$$\inf_{v_t} \frac{1}{2} \int_0^1 |v_t|_V^2 dt + \mathcal{D}(I_S \circ (\varphi^{v_t})^{-1}, I_T) \tag{2.5}$$

where $|\cdot|_V^2$ is a pre-Hilbertian norm on a subspace $V$ of the space of vector fields $\Gamma(\mathbb{R}^d)$ of $\mathbb{R}^d$. Several choices for the space $V$ have been studied leading to right-invariant structures on the group of diffeomorphisms $\mathrm{Diff}(\mathbb{R}^d)$.

For example, a vast amount of work has been dedicated to study right-invariant Riemannian metrics on the group of diffeomorphisms, and their properties, particularly because they appear naturally in mathematical fluid dynamics and geometric mechanics (see for instance [10, 32, 49, 20, 75, 56]). In these approaches, we consider groups of smooth diffeomorphisms, with some regularity conditions at infinity, in order to model these groups on functional spaces with good topological properties (cf. [70]). The Riemannian metrics are then *weak*, in the sense that the topology induced by the Riemannian structure is weaker than the intrinsic original topology of the diffeomorphism groups (see [14] for examples of such metrics).

In this work we consider another approach, that also leads to right-invariant structures. We introduce a Hilbert space $V$ induced by a *reproducible kernel* [40, 39]. The advantage of such a kernel is that it enables very efficient computations of many quantities involved in the minimizing problem (1.5). The resulting structure, however, is not Riemannian but only sub-Riemannian as described in [8, 5]. This is due to the fact that the group obtained by integrating the vector fields in $V$ does not carry in general a natural and usable differentiable structure. In [8, 5], authors expose a generalization to the infinite dimension of sub-Riemannian geometry (cf. [73]), with particular applications on the groups of diffeomorphisms. This allow a geometric description of LDDMM methods. This thesis builds upon this framework. In the following section, we summarize the main results obtained for these right-invariant structures on diffeomorphism groups.

### 2.2.2 Sub-Riemannian structures and geometric formulation

Let $M$ be a finite dimensional manifold, with bounded geometry (for example $M = \mathbb{R}^d$). Let $s \geq d/2 + 1$, and denote by $H^s(M, M)$ the space of $H^s$ mappings of $M$ (see for instance [32, 33, 80] for study of its differential structure). We then define the group of Sobolev diffeomorphisms

$$\mathrm{Diff}_{H^s}(M) = \mathrm{Diff}_{C^1}(M) \cap H^s(M, M)$$

It is a well-known fact that this group is a Hilbert manifold and a topological group [53]. Furthermore we also have

$$T_\varphi \mathrm{Diff}_{H^s}(M) = \Gamma_{H^s}(TM) \circ \varphi, \quad \varphi \in \mathrm{Diff}_{H^s}(M)$$

where $\Gamma_{H^s}(TM)$ is the set of $H^s$ vector fields of $M$.

Following [8], we can define right-invariant *strong* sub-Riemannian structures on $\mathrm{Diff}_{H^s}(M)$ as triples $(\mathrm{Diff}_{H^s}(M), V, \langle \cdot, \cdot \rangle_V)$, with $(V, \langle \cdot, \cdot \rangle_V)$ a Hilbert space (for example a RKHS) that is continuously embedded in $\Gamma_{H^s}(TM)$. This structure is associated with a vector bundle $\mathcal{H} \subset T\,\mathrm{Diff}_{H^s}(M)$ given by

$$\mathcal{H}_\varphi = V \circ \varphi \subset T_\varphi \mathrm{Diff}_{H^s}(M)$$

and a strong metric on this bundle

$$\langle v \circ \varphi, v \circ \varphi \rangle_\varphi = \langle v, v \rangle_V, \quad \forall \varphi \in \mathrm{Diff}_{H^s}(M),\ v \circ \varphi \in \mathcal{H}_\varphi$$

Compared to usual Riemannian geometry, the main difficulty is that we thus need to restrict ourselves to curves in $\mathrm{Diff}_{H^s}(M)$ with temporal derivative in this subbundle, or equivalently with the Eulerian derivative belonging to $V$. This leads to several difficulties and questions, such as controllability (exact or approximate), or results on the different completeness properties (metric completeness, geodesic connectedness, existence of solutions for the geodesic equations). A significant amount of work has been dedicated to studying these theoretical properties, we refer to [8, 7] for example.

Moreover, if we suppose now that the group of diffeomorphisms $\mathrm{Diff}_{H^s}(\mathbb{R}^d)$ acts on a shape space $\mathcal{Q}$, under some regularity conditions [6, 9], we can then induce these sub-Riemannian structures on $\mathcal{Q}$. This allows to establish geometric formulation of the minimization problem (1.5). In particular, these approaches yield characterization results for the minimizers and critical points of equation (1.5).

### 2.2.3 Extending the LDDMM framework

The formalism we presented above allows to obtain rich deformations and thereby solve many registration problems. Several extensions have then been developed to address more complex settings and achieve more accurate matchings. In particular, the metamorphosis framework [94, 50] for image matching introduces an additional term accounting for intensity variation within the deformation, thus enabling registration between images that may differ in topology [35, 51]. The deformation group considered in this case is a semidirect product of the diffeomorphism group with a functional space modeling the intensity changes, on which a sub-Riemannian metric is also defined.

More recently, some multiscale methods have also been developed in order to perform registrations with several levels of accuracy simultaneously [18, 84, 89, 71]. In these methods, we consider a RKHS generated by a sum of kernels, each kernel representing a different scale. Building on this idea, in [71] the authors consider that the shape space can also be expressed as a product of manifolds, allowing a shape to be represented at multiple scales. They then introduce a coarse-to-fine registration method, incorporating interactions between the different scales.

These various new problems thus highlight the limitations of the LDDMM framework for registration, since it relies solely on the action of a single diffeomorphism group. In this manuscript we aim to generalize and extend this framework to general infinite dimensional groups.

## Lie groups theory in infinite dimensions

Let us now delve more deeply into the topic of infinite dimensional groups. As mentioned previously, it is a rich subject and has to led to a wide literature, particularly because its numerous applications in physics and more specifically to geometric hydrodynamics [10]. Notably, this vast work has revealed phenomena that can occur only in infinite dimensions.

In particular, it is often necessary to consider topological vector spaces beyond the Banach setting when working in infinite dimensional geometry and Lie groups theory. We state as an example a theorem due to Omori [78].

**Theorem 2.1** ([78]). *Let $G$ be a a connected Banach–Lie group $G$ acting effectively, transitively and smoothly on a compact manifold. Then $G$ must be a finite-dimensional Lie group.*

Yet the large deformation framework for shape registration requires a group actino on a manifold, which is often finite dimensional. This implies, if we wish to work with Lie groups that it is necessary to consider Fréchet functional spaces (or even more general locally convex topologies), which can introduce analytical difficulties since the classical theorems such as the Picard-Lindelof theorem, local inversion, and others may no longer hold (see, for example, [79] for a study of certain types of infinite-dimensional Lie groups).

Another possibility would be to stay in the Banach setting and accept to work with groups with group structure and differentiable structure only partially compatible. In this thesis, we choose to work in the setting of Banach half-Lie groups [63, 16], which are topological groups and Banach manifolds such that the differentiability of the composition is only partial. However, we remain within the framework of Banach differential calculus. In particular, the groups of Sobolev diffeomorphisms, with regularity $H^s$ is an example of half-Lie groups, and in general many groups of diffeomorphisms with finite regularity are also half-Lie groups but not Lie groups.

## 2.3 Outline of the thesis

The aim of this thesis is to extend and study these right-invariant sub-Riemannian structures on Banach half-Lie groups. In particular, we intend to recover the main completeness results and to characterize normal geodesics (see [73]) through Hamiltonian equations. Moreover, we wish to induce these structures on classic shape spaces (such as curves, surfaces, images, landmarks) via group actions, and thereby extend the framework presented in [5, 6, 8]. We present several applications to matching problems and shape analysis.

This thesis primarily focuses on the theoretical constructions and the geometric understanding of shape spaces, as well as on the actions that can be defined on them. A few simple and minimal numerical examples are given to test these models. However, this work is devoted almost exclusively to the development of theoretical tools and results. This means further developments would be required for numerical applications, especially in medical imaging, involving more extensive studies and modeling efforts.

This thesis is organized into three parts, progressing from the most abstract to the most applied. First part (chapters 3 and 4) presents general calculus and results on Banach calculus, and recalls in particular properties of Banach half-Lie groups. Then second part (chapters 5 and 6) focuses on right-invariant sub-Riemannian geometry, and the definition of shape spaces we use in this manuscript. Finally we present some applications of this setting for shape analysis in chapters 7, 8 and 9. The thesis is arranged as follows:

**Chapter 3 : Calculus in Banach setting** The main references for this chapter are [61, 41, 80]. We recall here general results of Banach differential calculus, reviewing some important theorems such as implicit and inverse function theorems. We also present the theory of Bochner and strong integration in Banach spaces [91, 88], introducing in particular the space of absolutely continuous curves of a Banach manifold [41]. We prove this space of curves also forms a Banach manifold, and we explicit its differential structure.

**Chapter 4 : Banach half-Lie groups** In this chapter, we recall the definition of half-Lie groups [63, 16] and we review some basic properties and constructions associated with half-Lie groups. We discuss $L^p$-regularity of half-Lie groups, i.e. whether we can integrate $L^p$ curves in the tangent space at the identity to absolutely continuous curves in the group. Finally, we adapt results from Lie–Poisson reduction theory to the setting of right-invariant Hamiltonians on half-Lie groups.

**Chapter 5 : Right-invariant sub-Riemannian geometry on half-Lie groups** This chapter studies strong right-invariant sub-Riemannian geometry on half-Lie groups. Under some differentiability conditions, we prove some completeness properties of such structures, and we derive a sub-Riemannian Euler-Arnold-Poincaré equations describing the sub-Riemannian (normal) geodesics.

**Chapter 6 : Shape spaces and induced metrics** In this chapter, we define shape spaces as Banach manifolds acted upon by a half-Lie group, subject to some differentiability conditions. In particular, if the half-Lie group is equipped with a strong right-invariant sub-Riemannian structure, we can induce a corresponding right-invariant metric on the shape space. We then formulate a variational problem for inexact matching between shapes, and we study how the regularity of the covectors is conserved in the Hamiltonian dynamic.

**Chapter 7 : Multiscale matching** We present a first application to multiscale matching for shape analysis. We first introduce a general theoretical framework for coupling the classic large deformation model [95, 93] with the action of a finite-dimensional Lie group. We consider the example of rigid motions represented by the group of isometries, associated with the non rigid deformations induced by LDDMM. Then, following [71], we introduce hierarchical scheme allowing a multiscale registration of shape spaces.

**Chapter 8 : Anisotropic Gaussian kernel and scaling** This chapter describes the main results of [74], (joint work with PhD student Rayane Mouhli). We incorporate anisotropic information, encoded via an anisotropic Gaussian kernel, into the LDDMM setting. We model how rigid motions and scalings transport these anisotropic features, and given numerical examples on matching of landmarks and images.

**Chapter 9 : Transport of Riemannian metrics of a compact manifold** This chapter presents a work in progress, and also models the incorporation of anisotropy in shape analysis through the transport of metrics. A framework is introduced to model transport of metrics by diffeomorphisms and by automorphisms. In particular study the differential structure of the group of $C^k$ automorphisms of the tangent bundle of a compact manifold, and prove it forms a half-Lie group. This group induces a natural action on the space of Riemannian metrics by pushforward. We introduce a variational problem using the Ebin metric and provide some numerical examples.

## 2.4 Notations

Throughout this thesis we will use the following notations:

- If $E$ is a Banach space, its norm will be denoted by $|\cdot|_E$.
- $L(E, F)$, where $E$ and $F$ are Banach spaces : space of continuous linear maps between $E$ and $F$. The induced norm is denoted by $\|\cdot\|_{L(E,F)}$ to emphasize the fact that we are using the operator norm.
- If $E$ is a Banach space, its topological dual will be denoted by $E^*$.
- $(m \mid v)$, with $m \in E^*, v \in E$, is the evaluation of the linear continuous form $m$.
- $\langle u, v\rangle_V$ : inner product in a Hilbert space $V$.
- If $V$ is a Hilbert space, we will denote $K_V : V^* \to V$ the inverse of the canonical Riesz isometry :
$$\langle K_V p, v\rangle_V = (p \mid v).$$
- If $c : I \to E$ is a differentiable curve in a Banach space $E$ (or a Banach manifold), we will use the notation $\dot{c}$ for the time derivative. In the case $E$ is a functional space, we will sometimes write $\partial_t c$ to emphasize the fact that we are taking a time derivative and not a space derivative. The notation $c_t$ will be often used to indicate the evaluation $c(t)$ (and never the time derivative).

# Chapter 3

# Mathematical background I : Calculus in Banach spaces

## Contents



This chapter is devoted to the general calculus in the Banach setting. We define here the necessary concepts and tools that we will use in the rest of this work.

We start by recalling basic definitions and properties of Fréchet differential calculus in Banach spaces, and in particular some facts of the theory of differential geometry following [79, 61]. We give a definition of Vector bundles in this Banach setting and give the example of the cotangent bundle that will be needed for the Hamiltonian formulations given later. Most results here are given without proof and we refer mainly to the books of Lang and Palais [61, 80].

In section 3.2, we then briefly review the theory of strong integration, or Bochner integration in the Banach spaces [88, 91]. This allows to define in particular the space of absolutely continuous curves in Banach spaces (and Banach manifolds), following the work of Glockner [41]. We give an explicit construction of a Banach differentiable structure for the space of absolutely continuous curves. An analogous differential structure was also given recently in [83], using classic tools of infinite dimensional differential geometry. The construction we propose here is quite natural and follows from [60]. It allows in particular to describe Carathéodory differential equations, and we also enunciate the Picard-Lindelof theorem.

Finally, we end the chapter with section 3.3, which describes a class of variational problems from optimal control theory. We particularly describe the critical points for such problems. We refer to [3, 96] for general theory of optimal control theory (in particular for finite dimensional problems).

## 3.1 Differentiable calculus in Banach spaces

In this part, we introduce basic definitions and properties on differentiable calculus in Banach spaces. This will allow to specify and explain notations that we will encounter throughout this thesis, and to define a proper setting for the next chapters. Most of the definitions and results can be found in [79, 61], with more detailed explanations

### 3.1.1 Definitions and basic properties

#### 3.1.1.1 Frechet differential calculus

We introduce here the first definitions of Frechet calculus. Let $E, F$ be Banach spaces, $U \subset E$ an open subset of $E$, and $f : E \to F$ a continuous function. We recall here the definition of differentiability.

**Definition 3.1** (Frechet differentiability)**.** We say that $f$ is *differentiable* (or Fréchet differentiable) at $x \in U$, if there is an element $A_x \in L(E, F)$ such that for any $\delta x \in E$ close enough to $x$,

$$f(x + \delta x) = f(x) + A_x \delta x + o(|\delta x|_E)$$

We denote by $d_x f$ the linear map $A_x$.

**Remark 3.2.** *Note that this definition is stronger than the usual Gateaux differentiability that we recall here: the map $f$ is said to be* Gateaux differentiable *at $x \in U$, if for any $\delta x \in E$, the curve $t \mapsto f(x + t\delta x)$ is differentiable at $t = 0$. We denote by $df(x; \delta x)$ this directional derivative. In particular, if $f$ is (Fréchet) differentiable at $x$, then it is Gâteaux differentiable at $x$.*

If $f$ is differentiable at any $x \in U$, this defines a mapping $df : U \to L(E, F)$, called the differential or the derivative of $f$. In particular, we can define $C^1$, and inductively $C^k$ differentiability.

**Definition 3.3** ($C^k$ differentiability)**.** The map $f$ is said to be $C^1$ if $f$ is differentiable at any $x \in U$, and the differential $df : U \to L(E, F)$ is continuous. Inductively, we then say that $f$ is $C^k$, for $k \geq 1$, if $df$ is $C^{k-1}$, and that $f$ is smooth, or $C^\infty$ if $f$ is $C^k$ for all $k \in \mathbb{N}$.

**Remark 3.4.** *Note that if $f$ is a $C^k$, its $k$-th derivative $d^k f$ is canonically identified with an element of $L^k_{sym}(E, ..., E; F)$, the space of symmetric $k$-linear maps. Moreover, as usual, sums, compositions and classic operation on $C^k$ maps give $C^k$ maps.*

The space of $C^k$ maps is simply denoted by $C^k(E,F)$. As we saw, Fréchet differentiability implies Gâteaux differentiability. We recall a sufficient condition for the converse:

**Lemma 3.5** (From Gâteaux to Fréchet)**.** *Suppose that $f : U \to F$ is Gâteaux differentiable at any $x \in U$ and that for any $x \in U$, the derivative $df(x;\cdot)$ is linear and continuous, and that the induced map*

$$\begin{array}{ccc} U & \longrightarrow & L(E,F) \\ x & \longmapsto & df(x;.) \end{array}$$

*is continuous. Then the map $f$ is $C^1$, and $d_x f = df(x;.)$*

*Proof.* Let $x \in U$, and $\delta x \in E$, close enough to $x$. Then we get, by definition

$$\begin{aligned} f(x+\delta x) - f(x) - df(x;\delta x) &= \int_0^1 \partial_t f(x+t\delta x) - df(x;\delta x)dt \\ &= \int_0^1 df(x+t\delta x;\delta x) - df(x;\delta x)dt \end{aligned}$$

In particular, since $x \mapsto df(x;.)$ is continuous, if the norm of $\delta x$ is small enough, then there exists a constant $c > 0$ such that, for all $t \in [0,1]$,

$$\|df(x+t\delta x;\delta x) - df(x;\delta x)\|_{L(E,F)} \leq c|\delta x|_E,$$

which finishes to prove the result. $\square$

We end this part by defining the notion of partial derivative. Let $E, F, G$ be Banach spaces, $U \subset E, V \subset F$ open subset. We consider $f : U \times V \to G$ a continuous functions, and $(x,y) \in U \times V$. Suppose we keep $y$ fixed, and that the map

$$\begin{array}{ccc} U & \longrightarrow & G \\ x & \longmapsto & f(x,y) \end{array}$$

is differentiable. We denote its derivative by $\partial_x f(x,y) \in L(E,G)$, and we call it the *partial derivative* of $f$ with regards to the first variable. In particular, if the partial derivative exists for all $(x,y) \in U \times V$, the partial derivative

$$\partial_x f : \begin{array}{ccc} U \times V & \longrightarrow & L(E,G) \\ (x,y) & \longmapsto & \partial_x f(x,y) \end{array}$$

defines a map from $U \times V$ to $L(E,G)$. The next proposition relates the differential of a map to its partial derivatives, the proof can be found in [61].

**Proposition 3.6** (Partial derivative)**.** *Let $f : U \times V \to G$ be a continuous functions. Then $f$ is $C^p$ if and only if for any $(x,y) \in U \times V$, the partial derivatives exist and are $C^{p-1}$. In such case, we have for any $(x,y) \in U \times V$, $(\delta x, \delta y) \in E \times F$*

$$df(x,y)(\delta x, \delta y) = \partial_x f(x,y)\delta x + \partial_y f(x,y)\delta y.$$

#### 3.1.1.2 Inverse function theorem

In this section, we recall the inverse function theorem, and the implicit function theorem in Banach manifolds. Let $E, F, G$ be Banach spaces, and $U \subset E$, $V \subset F$ open subsets. We first recall the definition of a diffeomorphism.

**Definition 3.7** (Diffeomorphism)**.** A $C^k$ (resp. smooth) bijective map $f : U \to V$ is called a $C^k$ diffeomorphism (resp. a smooth diffeomorphism) if its inverse $f^{-1}$ is also $C^k$ (resp. smooth).

Now we can state the inverse function theorem

**Theorem 3.8** (Inverse function theorem)**.** *Let $f : U \to F$ be a $C^1$ mapping, $x \in U$, and suppose that the derivative $d_x f$ is invertible and continuous. Then $f$ is a local diffeomorphisms from a neighborhood of $x$ onto a neighborhood of $f(x)$.*

**Remark 3.9.** *Note that, since $E$ and $F$ are Banach spaces, if $d_x f$ is continuous and invertible, then the inverse $d_x f^{-1}$ is also continuous by Banach isomorphism theorem.*

We also recall the implicit function theorem that we will use a lot in this thesis

**Theorem 3.10** (Implicit function theorem)**.** *Let $f : U \times V \to G$ a $C^k$ mapping, $(x_0, y_0) \in U \times V$ such that $f(x_0, y_0) = 0$ and such that the partial differential*

$$\partial_y f(x_0, y_0) : F \to G$$

*is invertible with continuous inverse. Then there exists a $C^k$ map defined on a neighborhood $U_0 \subset U$ of $x_0$, such that $g(x_0) = y_0$ and*

$$f(x, g(x)) = 0$$

*for all $x \in U_0$.*

#### 3.1.1.3 Manifolds, submersions, immersions

We recall here the definition of a Banach manifold, with its charts. In this thesis, all the manifolds, unless stated otherwise, will be smooth manifolds so we only use this definition.

**Definition 3.11** (Manifold, charts)**.** Let $\mathbb{B}$ be a Banach space. A Hausdorff space $M$ is called a (smooth) *manifold*, modeled on $\mathbb{B}$, if there exists an indexed collection of pairs $(U_\alpha, \phi_\alpha)_{\alpha \in A}$, such that :

- Each $U_\alpha$ is an open subset of $M$, and the family $(U_\alpha)$ defines a covering of $M$.
- Each $\phi_\alpha$ is a bijection from $U_\alpha$ to an open subset $\phi_\alpha(U_\alpha)$ of $\mathbb{B}$.
- For any $\alpha, \alpha' \in A$ the set $\phi_\alpha(U_\alpha \cap U_{\alpha'})$ is an open subset of $\mathbb{B}$, and the map

$$\phi_{\alpha'} \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_{\alpha'}) \to \phi_{\alpha'}(U_\alpha \cap U_{\alpha'})$$

  is a smooth diffeomorphism.

In such case, each pair $(U_\alpha, \phi_\alpha)$ is called an *open chart* of the manifold $E$, and the collection $(U_\alpha, \phi_\alpha)$ is called an *atlas* of $M$.

**Remark 3.12.** *Note that in this definition we just use one atlas of $M$. The definition of a structure of smooth manifold is actually the class of all compatible atlases, where the compatible relationship is defined using the third condition of the definition of the atlas [61]. We will not focus too much on these technicalities, since defining manifold structures on spaces just requires defining an atlas. Moreover, we can associate to any manifold $M$ its usual* tangent bundle $TM$, *that can be built using equivalence classes of smooth curves in $M$. We will not dwell on this construction either and refer to [61].*

**Remark 3.13.** *We will often require the Banach space $E$ to be also separable, in particular to define strong integration and absolutely continuous curves. Most of the examples seen in this thesis will be modeled on separable Banach spaces.*

This definition allows to do differentiable calculus in spaces that locally look like Banach vector spaces. In particular, let $f : M \to N$ be a continuous map between smooth manifolds $M$ and $N$. Then the map $f$ is said to be differentiable, $C^k$ or smooth, if it is the case in the local charts of $M$ and $N$, meaning that for every chart $(U_\alpha, \phi_\alpha)$ of $M$ and $(V_{\alpha'}, \phi_{\alpha'})$ of $N$, such that $f(U_\alpha) \subset V_{\alpha'}$, the map $\hat{f} = \phi_{\alpha'} \circ f \circ \phi_\alpha^{-1} : U_\alpha \to V_{\alpha'}$ is differentiable, $C^k$ or smooth. In this case, for any $x \in M$ we can define the differential of $f$ at $x$, or the tangent mapping of $f$, that we denote $T_x f$ which corresponds in local charts of $M$ and $N$ to the differential of $\hat{f}$ we defined in the previous section. We recall also the definition of immersions and submersions, in the sense of [61] as we will require another assumptions on these maps.

**Definition 3.14** (Submersions, immersions, submanifold)**.** Let $M$, $N$ be smooth Banach manifolds, and $f : M \to N$ a $C^1$ map, and $x \in M$. Then

- The map $f$ is an *immersion* (or *splitting immersion*) at $x \in M$ if the derivative $T_x f : T_x M \to T_{f(x)} N$ is injective and splits, meaning that the vector space $T_x f(T_x M)$ is a closed subspace of $T_x M$ and admits a closed supplementary.
- The map $f$ is a *submersion* (or *splitting submersion*) at $x \in M$ if the derivative $T_x f : T_x M \to T_{f(x)} N$ is surjective and its kernel splits, meaning that its kernel admits a closed supplementary in $T_x M$.
- The map $f$ is a *diffeomorphism* if it is bijective, and its inverse $f^{-1}$ is differentiable.

**Remark 3.15.** *Note that if $M$ and $N$ are finite dimensional manifolds, then the splitting conditions in the definition are automatically satisfied. These are necessary conditions to describe locally immersions and submersions as local embeddings or local projections.*

In the next proposition, we describe locally immersions and submersions and retrieve a result of finite dimensional geometry

**Proposition 3.16** (Local description of immersions and submersions)**.** *Let $f : M \to N$ a $C^1$ map between Banach manifolds $M$ and $N$, and let $x \in M$,*

- *If $f$ is an immersion at $x$, then there exists a chart $(U, \phi)$ at $x$, a chart $(V, \psi)$ at $f(x)$ such that $\psi$ gives a decomposition of $V$ as product $\tilde{V}_1 \times \tilde{V}_2$ with $V_1$ and $V_2$ open subset and such that*

$$\tilde{f} = \psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) = \tilde{V}_1 \times \tilde{V}_2$$

*is an isomorphism from $\phi(U)$ to $V_1$.*

- *If $f$ is a submersion at $x$, then there exists a chart $(U, \phi)$ at $x$, a chart $(V, \psi)$ at $f(x)$, such that $\phi$ gives a decomposition of $U$ as product $\tilde{U}_1 \times \tilde{U}_2$ with $\tilde{U}_1$ and $\tilde{U}_2$ open subset and such that*

$$\tilde{f} = \psi \circ f \circ \phi^{-1} : \tilde{U}_1 \times \tilde{U}_2 \to \psi(V)$$

*is a projection.*

### 3.1.2 Vector bundles and morphisms

Vector bundles are important objects of differential geometry, and the definition in the Banach setting require specific additional hypotheses compared to the finite dimensional case. In this section, we give the definition of Banach vector bundles and vector bundles morphisms as defined in [61]. Let $M, E$ be Banach differentiable manifolds, and $\mathbb{B}_E$ be a Banach space. We are also given a $C^k$ map $p : E \to M$. We give conditions on $p$ to define a vector bundle on $E$ over $M$.

**Definition 3.17** (Banach vector bundle)**.** Suppose we are given an open covering $(U_\alpha)_{\alpha\in A}$ of $M$, and smooth maps $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{B}_E$ such that

1. The map $\phi_\alpha$ is a $C^k$ diffeomorphism and we have the commutative diagram

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "p"'] & & U_\alpha \times \mathbb{B}_E \arrow[dl, "\pi_1"] \\ & U_\alpha & \end{tikzcd}$$

   We get in particular an isomorphism on each fiber, that we denote by

$$\phi_{\alpha x} : p^{-1}(x) \to \{x\} \times \mathbb{B}_E,$$

   where $x \in M$.

2. For any $U_\alpha, U_{\alpha'}$ and $x \in U_\alpha \cap U_{\alpha'}$, the mapping

$$\phi_{\alpha x} \circ \phi_{\alpha' x}^{-1} : \mathbb{B}_E \to \mathbb{B}_E$$

   is a continuous linear isomorphism with continuous inverse.

3. For any $U_\alpha, U_{\alpha'}$ the map

$$x \mapsto \phi_{\alpha x} \circ \phi_{\alpha' x}^{-1}$$

   is $C^k$ mapping from $U_\alpha \cap U_{\alpha'}$ onto $L(\mathbb{B}_E, \mathbb{B}_E)$.

In such case, the family of $(U_\alpha, \phi_\alpha)$ defines a $C^k$ Banach vector bundle structure : the triple $(E,M,p)$ is called a $C^k$ *vector bundle*. Each $(U_\alpha, \phi_\alpha)$ is called a *trivialization* of this vector bundle, and the morphism $\phi_\alpha$ is the *trivializing map*. The manifold $E$ is called the *total space* and $M$ the *base space*.

**Remark 3.18.** *Note that the third point is not necessary in finite dimensions because it is directly implied by the second condition. Moreover, we will see it is also implied by the second condition if we replace $C^k$ by smooth, which defines a smooth vector bundle structure.*

#### 3.1.2.1 Vector bundle morphisms

We next define the morphisms of a vector bundle, that is to say morphisms that are compatible with the structure defined in 3.17 (and in particular with the third condition). We start by recalling the notion of pullback of a vector bundle. Let $p : E \to N$ denote a vector bundle over a Banach manifold $N$. Let $M$ be a Banach manifold, and $f : M \to N$ a $C^k$ map. Then we define the vector bundle $f^*E$ over $M$, called the *pullback* of $E$, as the triple $f^*E = (E, M, p \circ f)$. The fiber at $x \in M$ is given by

$$(f^*E)_x = E_{f(x)}$$

and in particular, we get a vector bundle morphism

$$\begin{array}{ccc} f^*E & \xrightarrow{\mathrm{id}} & E \\ \big\downarrow{\scriptstyle p\circ f} & & \big\downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & N \end{array}$$

We next define vector bundle morphisms following Lang [61].

**Definition 3.19** (Vector bundle morphism)**.** Let $p_E : E \to M$, $p_F : F \to N$ be $C^k$ vector bundles, and let $\underline{f} : M \to N$ and $f : E \to F$ be a pair of $C^k$ maps such that

1. The map $f$ is *fiber-preserving* with regards to $\underline{f}$, that is to say the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \big\downarrow{\scriptstyle p_E} & & \big\downarrow{\scriptstyle p_F} \\ M & \xrightarrow{\underline{f}} & N \end{array}$$

   and the induced map on the fibers $f_x : E_x \to F_{\underline{f}(x)}$ is continuous and linear.

2. The induced map
$$\begin{array}{ccc} M & \longrightarrow & L(E, \underline{f}^*F) \\ x & \longmapsto & f_x \end{array}$$
   is $C^k$.

Then the pair $(\underline{f}, f)$ defines a $C^k$*-vector bundle morphism.*

**Remark 3.20.** *Note that, once again, the second condition is redundant in finite dimension.*

The second point can be tricky to verify in practice, and we often only have regularity of the maps $\underline{f}$ and $f$. The following result allows to get vector bundle morphisms from this first hypothesis only, but with a loss of degree of regularity.

**Proposition 3.21** (Regularity of fiber preserving morphisms)**.** *Let $E, F$ two vector bundles over $M$ and $N$ (with Banach differentiable structure). Suppose $f : E \to F$ is a $C^1$ mapping, fiber preserving and linear in the fibers. Then $f$ is a continuous locally Lispchitz vector bundle morphism. Moreover, if $f$ is $C^l$, then $f$ is a $C^{l-1}$ vector bundle morphism.*

*Proof.* The second part of the theorem is actually contained in [79, theorem 5.3] and proved by induction. We adapt the proof to show the first point. Suppose $f$ is $C^1$, fibered preserving and linear in the fibers. Let $\mathbb{B}_E$ (resp. $\mathbb{B}_F$) be a Banach space that models the fibers of $E$ (resp. $F$). Let $x_0 \in M$, and $(U, \tau_E)$ and $(V, \tau_F)$ trivializing charts such that $x_0 \in U$, and $f^{-1}(V) \subset U$. Let also $x \in U \to \widehat{f}_x \in L(\mathbb{B}_E, \mathbb{B}_F)$ be the local representation of $x \mapsto f_x$ in local trivializations we defined. In particular, by assumption the map $(x, u) \in U \times \mathbb{B}_E \mapsto \widehat{f}_x(u) = \widehat{f}(x, u)$ is $C^1$. Therefore, there exists $K, \delta > 0$ and $U_0$ open convex neighborhood of $x_0$ in $\mathbb{B}_E$, such that for any $x \in U_0$ and $u \in B_{\mathbb{B}_E}(0, \delta)$, we have

$$|\partial_1 \widehat{f}(x_0, u) - \partial_1 \widehat{f}(x, u)|_{L(\mathbb{B}_E, \mathbb{B}_F)} < K$$

Therefore we get, for $x, x' \in U_0$ and $u \in B_{\mathbb{B}_E}(0, \delta)$

$$\begin{aligned}
|\widehat{f}(x,u) - \widehat{f}(x',u)|_{\mathbb{B}_F} &\leq \int_0^1 \left| \partial_1 \widehat{f}\left((1-t)x + tx', u\right)(x'-x) \right|_{\mathbb{B}_F} dt \\
&\leq \int_0^1 \left| \partial_1 \widehat{f}\left((1-t)x + tx', \frac{u\delta}{|u|}\right)(x'-x) \right|_{\mathbb{B}_F} \frac{|u|}{\delta} dt \\
&\leq \int_0^1 K|x'-x|_{\mathbb{B}_E} \frac{|u|}{\delta} dt = K|x'-x|_{\mathbb{B}_E} \frac{|u|}{\delta}
\end{aligned}$$

Thus we finally get

$$|\widehat{f}(x,u) - \widehat{f}(x',u)|_{L(\mathbb{B}_E, \mathbb{B}_F)} < \frac{K}{\delta}|x'-x|_{\mathbb{B}_E}$$

and $x \mapsto f_x$ is locally Lipschitz. $\square$

#### 3.1.2.2 Some operations on vector bundles

We recall without proof some classic constructions of vector bundles. These examples can be found with more details in [61, chapter 3]. Naturally, if $M$ is a Banach manifolds, the collection of tangent spaces $TM$ is a vector bundle over $M$. We call *cotangent bundle* $T^*M$ the dual of the tangent bundle $TM$, with fibers at $x \in M$ given by

$$T_x^*M = L(T_xM, \mathbb{R})$$

More generally if $E \to M$ is a vector bundle over $M$, we can define the *dual bundle* $E^* \to M$, where for every $x \in M$, the fiber $E_x^* = L(E_x, \mathbb{R})$ is the topological dual of $E_x$. Similarly, if $r \in \mathbb{N}$, we can take the vector bundle $L^r(E)$ of multi-linear forms on $E$. If $F \to M$ is also a vector bundle over $M$, this induces the vector bundle $L(E, F) \to M$ of linear maps from $E$ to $F$, with fiber given at $x \in M$ by

$$L(E, F)_x = L(E_x, F_x).$$

Note that if $F \to N$ is a vector bundle over another Banach manifold $N$, we also denote by $L(E, F)$ the vector bundle over the direct product $M \times N$, with fiber at $(x, y) \in M \times N$ given by

$$L(E, F)_{x,y} = L(E_x, F_y)$$

We also define first the direct sum of vector bundles. Let $E \to M$ and $F \to M$ be two vector bundles over $M$ a Banach manifold. We define the *direct sum* $E \oplus F$ with fiber at $x \in M$ given by

$$(E \oplus F)_x = E_x \oplus F_x.$$

### 3.1.3 Cotangent bundle and symplectic geometry

We remind some basic definitions and properties of symplectic geometry on the cotangent bundle to have a better understanding of the notions involved in future sections. General symplectic geometry in infinite dimension is not developed in this section, we just introduce the basic definitions necessary to carry out the work we will do. We refer to [25] for a general exposition in finite dimension, and to [2, 65] for infinite dimensional spaces. Throughout this section, $\mathcal{Q}$ will denote a Banach manifold, and we consider the cotangent bundle $T^*\mathcal{Q}$. We start by the definition of the symplectic form on $T^*\mathcal{Q}$.

**Definition 3.22** (Weak symplectic form)**.** A 2-form $\omega$ on $T^*\mathcal{Q}$ is called a *weak symplectic form* if it is closed, and weakly non-degenerate, meaning that $d\omega = 0$ and for any $p \in T^*\mathcal{Q}$, the map $\delta p \mapsto \omega_p(\delta p, .)$ is a linear injective continuous map from $T_pT^*\mathcal{Q}$ to $T_p^*T^*\mathcal{Q}$.

We denote by $\omega^\flat : TT^*\mathcal{Q} \to T^*T^*\mathcal{Q}$ the *flat map* defined by $\delta p \mapsto \omega_p(\delta p, .)$. The symplectic form is called strong if the flat map is also onto and defines a vector bundle isomorphism. Since Banach spaces are in general not reflexive, we often have to deal with weak symplectic manifolds. We can equip the cotangent bundle $T^*\mathcal{Q}$ with a canonical weak symplectic form, defined as the exterior derivative of the Liouville form. In canonical local coordinates of $TT^*\mathcal{Q}$ around $(q, p) \in T^*\mathcal{Q}$, it is defined as

$$\omega_{q,p}\left((\delta q, \delta p), (\delta q', \delta p')\right) = \delta p(\delta q') - \delta p'(\delta q).$$

We suppose for the rest of the section that $T^*\mathcal{Q}$ is equipped with this canonical symplectic form. Contrary to the finite dimensional case, since the flat map $\omega^\flat$ is not onto, the symplectic gradient is not defined for any smooth functions

**Definition 3.23** (Hamiltonian vector field)**.** Let $H : T^*\mathcal{Q} \to \mathbb{R}$ be a smooth function. We say $H$ is a Hamiltonian on $T^*\mathcal{Q}$ if there exists a vector field $X_H$ on $T^*\mathcal{Q}$ such that $\iota_{X_H}\omega = dH$. In such case, the vector field $X_H$ is unique, and called the *Hamiltonian vector field* or the *symplectic gradient* of $H$, and is also denoted by $\nabla^\omega H$

If $H$ is a Hamiltonian function on $T^*\mathcal{Q}$, its symplectic gradient is expressed in the canonical coordinates as

$$\nabla^\omega H(q, p) = \left(\partial_p H(q, p), -\partial_q H(q, p)\right).$$

In particular, a smooth map $H : G \to \mathbb{R}$ admits a symplectic gradient if and only if the partial derivative $\partial_p H(q, p)$ belongs to $T_q\mathcal{Q}$, where we identify $T_q\mathcal{Q}$ to its image in $T_q^{**}\mathcal{Q}$ through the canonical injection $T\mathcal{Q} \hookrightarrow T^{**}\mathcal{Q}$.

#### 3.1.3.1 Hamiltonian actions of finite dimensional Lie groups

We finish these sections with some results and properties of Lie group actions on symplectic manifolds, and in particular reduction theory. We suppose here the manifold $\mathcal{Q}$ is finite dimensional. Reduction theory and Noether's theorem arise from the symmetry of the studied system. Those symmetries are represented by a Lie group $G$ acting smoothly on the symplectic manifold $T^*\mathcal{Q}$ via $\psi : G \to \mathrm{Diff}(T^*\mathcal{Q})$. In addition, we assume that $G$ acts by symplectomorphism on $T^*\mathcal{Q}$, i.e preserves the symplectic form.

**Definition 3.24** (Hamiltonian action)**.** The action $\psi$ is said to be a Hamiltonian action if there exists a map $\mu : M \to \mathfrak{g}^*$ such that

- For every $X \in \mathfrak{g}$, by denoting $\hat{\mu}(X) : p \mapsto (\mu(p)|X)$, the vector field $\tilde{X}$ on $M$ generated by the infinitesimal action satisfies

$$\nabla^\omega \hat{\mu}(X) = \tilde{X}$$

  i.e $\tilde{X}$ is the symplectic gradient of $\hat{\mu}(X)$.

- $\mu$ is G-equivariant :

$$\mu \circ \psi_g = Ad_g^* \circ \mu \text{ for all } g \in G$$

The Noether's theorem states that symmetries give rise to conserved quantities.

**Theorem 3.25** (Noether's theorem)**.** *If the Hamiltonian $H$ is $G-$invariant, then the momentum maps $\hat{\mu}(X) : p \in T^*\mathcal{Q} \mapsto (\mu(p)|X) \in \mathbb{R}$ are conserved quantities along the Hamiltonian flow.*

We denote the orbit of G through $(q,p) \in T^*\mathcal{Q}$ by $\text{Orbit}_G(q,p) = \{\psi_g(q,p) \mid g \in G\}$ and the stabilizer of $(q,p) \in T^*\mathcal{Q}$ by the subgroup $G_p := \{g \in G \mid \psi_g(q,p) = (q,p)\}$. Being in the same orbit defines an equivalence relation $(q,p) \sim (q',p')$ if and only if $(q,p)$ and $(q',p')$ are on the same orbit, which defined an orbit space $T^*\mathcal{Q}/G$. Then, we define the mapping

$$\pi : \begin{array}{ccc} M & \longrightarrow & M/G \\ q,p & \longmapsto & \text{Orbit}_G(q,p) \end{array}$$

called the point-orbit projection. We can equip $T^*\mathcal{Q}/G$ with the quotient topology, which is the weakest topology for which $\pi$ is continuous, i.e $\mathcal{U} \subset T^*\mathcal{Q}/G$ is open if and only if $\pi^{-1}(\mathcal{U})$ is open in $T^*\mathcal{Q}$.

**Theorem 3.26** (Marsden-Weinstein-Meyer [66, 65])**.** *Suppose $(T^*\mathcal{Q}, \omega)$ is a finite dimensional symplectic manifold and let $G$ be a Lie group with Hamiltonian action on $(T^*\mathcal{Q}, \omega)$ with associated moment map $\mu : T^*\mathcal{Q} \to \mathfrak{g}^*$. Let $i : \mu^{-1}(0) \hookrightarrow T^*\mathcal{Q}$ be the inclusion map. Assume that $G$ acts freely and properly on $\mu^{-1}(0)$. Then,*

- *the orbit space $\mu^{-1}(0)/G$ (sometimes denoted $T^*\mathcal{Q}//G$) is a symplectic manifold.*
- *$\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principal G-bundle*
- *there is a symplectic form $\omega_{red}$ on $\mu^{-1}(0)/G$ satisfying $i^*\omega = \pi^*\omega_{red}$.*

*Moreover, if the action of $G$ on $T^*\mathcal{Q}$ is the cotangent lift of an action of $G$ on $\mathcal{Q}$, then we get*

$$T^*\mathcal{Q}//G \simeq T^*(\mathcal{Q}/G)$$

The pair $(\mu^{-1}(0)/G, \omega_{red})$ is called the reduced space (or symplectic quotient).

### 3.1.4 Manifold of mappings

Manifolds of mappings with finite regularity are very important examples of Banach manifolds, in particular in shape analysis as they are used to construct manifolds of shapes. They were particularly developed in [80, 34]. We give some examples here without giving their differential structures:

**Example 3.27** (continuously differentiable functions in $\mathbb{R}^d$)**.** *Let $k \geq 0$, then the space $C_0^k(\mathbb{R}^d, \mathbb{R}^d)$ of $C^k$ functions of $\mathbb{R}^d$ that vanish at infinity is a Banach vector space, and therefore directly a Banach manifold. It's the model vector space that we will use in many examples in the next sections.*

**Example 3.28** (continuously differentiable functions of compact manifolds)**.** *Let $k \geq 0$, $M$ be a compact manifold and $N$ be a finite dimensional manifold. The space $C^k(M, N)$ of $C^k$ mappings from $M$ to $N$ is a smooth Banach manifold. A proof of this result can be found in [67]. This can also be directly seen as an application of the next example.*

**Example 3.29** ($C^k$ sections of a fiber bundle)**.** *Let $k \geq 0$, $M$ be a compact manifold of dimension $d \in \mathbb{N}$ and $p : E \to M$ a smooth fiber bundle over $M$, with $E$ of dimension $n \in \mathbb{N}$. Denote by $\Gamma_{C^k}(E)$ the space of $C^k$ sections of $E$. Then, the space $\Gamma_{C^k}(E)$ is a Banach manifold [80, 34]. We explain how a differentiable structure can be built on this space. Let $s \in \Gamma_{C^k}(E)$. Then there exists [80, 12.6] a vector bundle neighborhood $\xi$ of $s$ in $\Gamma_{C^k}(E)$, i.e. the manifold $\xi$ is a subbundle of $E$ over $M$ such that $\xi$ is open in $E$, $s \in \Gamma_{C^k}(\xi)$ and that can be equipped with a vector bundle structure over $M$. Therefore the space $\Gamma_{C^k}(\xi)$ inherits of a Banach vector space structure, and the collection of such $\Gamma_{C^k}(\xi)$ form an atlas of a Banach differential manifold structure on $\Gamma_{C^k}(E)$. Moreover, the compact manifold $M$ can be covered by a finite number of compact submanifolds $M_1, ..., M_l$, with $l \in \mathbb{N}$, such that each vector bundle $\xi \to M_i$ is a trivialization, i.e. we have the following commutative diagram*

$$\begin{array}{ccc} \xi|_{M_i} & \xrightarrow{\phi_i} & \bar{B}_1 \times \mathbb{R}^n \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \pi_1} \\ M_i & \xrightarrow{\underline{\phi_i}} & \bar{B}_1 \end{array}$$

*where $\bar{B}_1$ is the closed unit ball of $\mathbb{R}^d$ and $(\underline{\phi_i}, \phi_i)$ is a smooth fiber bundle isomorphism. The Banach space $\Gamma_{C^k}(\xi)$ can then be identified [80, §.4] to the space*

$$\Gamma_{C^k}\left(\xi; (M_i, \phi_i)_i\right) = \{(f_1, ..., f_l) \in \prod_{i \leq l} C^k(\bar{B}_1, \mathbb{R}^n) \mid \phi_i^{-1} f_i \underline{\phi_i}_{|M_i \cap M_j} = \phi_j^{-1} f_j \underline{\phi_j}_{|M_i \cap M_j}\},$$

*which is a closed vector subspace of the Banach space $\prod_{i \leq l} C^k(\bar{B}_1, \mathbb{R}^n)$, through the smooth isomorphism*

$$\psi : \left\{ \begin{array}{ccc} \Gamma_{C^k}(\xi) & \longrightarrow & \Gamma_{C^k}\left(\xi; (M_i, \phi_i)_i\right) \\ \Gamma & \longmapsto & (\phi_1 \Gamma \underline{\phi_1}^{-1}, ..., \phi_k \Gamma \underline{\phi_k}^{-1}) \end{array} \right.$$

*In particular isomorphism gives a description of the charts of $\Gamma_{C^k}(E)$ as a subspace of a product of continuously differentiable functions, where it is easier to do calculus.*

## 3.2 Integration and ordinary differential equations

In this section, we introduce the theory of strong, or Bochner integration in Banach spaces, and how we can integrate differential equations with only $L^p$ regularity in time. In particular, we define the space of absolutely continuous curves in Banach spaces and Banach manifolds, and we prove we can equip this space with a differential structure.

### 3.2.1 Strong integration and $L^p$ spaces

We define here brieftly the strong integration for Banach-valued functions. We refer to [91, 88] for a more complete exposition. In this section, $\mathbb{B}$ denotes a Banach space, and $I$ is a bounded open interval of $\mathbb{R}$. We recall first the notion of strong measurability. A function $f : I \to \mathbb{B}$ is called *simple* if there exists a finite number $(E_i)_{1 \leq i \leq n}$ of measurable subsets of $I$ such that

$$f = \sum_{i=1}^{n} 1_{E_i} b_i$$

where $b_i \in \mathbb{B}$.

**Definition 3.30** (Strong measurability). A map $f : I \to \mathbb{B}$ is said to be *strongly measurable*, or *Bochner measurable* if there exists a sequence of simple functions $f_n$ that converges pointwise to $f$

We also recall the Pettis theorem that provides a useful characterization of strongly measurable maps.

**Theorem 3.31** (Pettis measurability theorem). *Let $f : I \to \mathbb{B}$, the following assertions are equivalent*

1. *$f$ is strongly measurable*
2. *$f$ is separably valued and for any $l \in \mathbb{B}^*$, the map $l \circ f : I \to \mathbb{R}$ is measurable (in the classic sense)*

**Remark 3.32.** *A map $f : I \to \mathbb{B}$ that verifies that $l \circ f$ is measurable for any $l \in \mathbb{B}^*$ is called* weakly measurable. *We will not focus too much on this notion in this thesis.*

In particular, continuous functions from $I$ to $\mathbb{B}$ are strongly measurable. Moreover, if $g : \mathbb{B} \to \mathbb{B}'$ is a continuous map from Banach spaces and $f : I \to \mathbb{B}$ is stongly measurable, then the composition $g \circ f : I \to \mathbb{B}'$ is also strongly measurable. We next recall the strong integrability and the integral of functions. If $f = \sum_{i=1}^{n} 1_{E_I} b_i$ is simple, with $E_i \cap E_j = \emptyset$ for any $i, j \in \{1, .., n\}$, we define its integral simply by

$$\int_I f(t)dt = \sum_I b_i \lambda_1(E_I)$$

where $\lambda_1$ is the usual Lebesgue measure on $\mathbb{R}$.

**Definition 3.33** (Bochner integral)**.** A map $f : I \to \mathbb{B}$ is called *Bochner integrable* if $f$ is strongly measurable and

$$\int_I |f(t)|_{\mathbb{B}} dt < \infty.$$

Equivalently, $f$ is Bochner integrable if and only if $f$ is a limit of simple functions $f_n$ [52, 1.2.2] and

$$\lim_{n\to\infty} \int_I |f(t) - f_n(t)|_{\mathbb{B}} dt = 0.$$

We define then the integral of $f$ as

$$\int_I f(t)dt = \lim_{n\to\infty} \int_I f_n(t)dt$$

and this definition is independent of the sequence $f_n$. We finish this part by introducing the space of $L^p$ functions from $I$ to $\mathbb{B}$. We recall that two functions $f : I \to \mathbb{B}$ and $g : I \to \mathbb{B}$ are said to be equivalent if $f(t) = g(t)$ for almost every $t \in I$. This defines an equivalence relation on the set of strongly measurable functions, and we will work in the quotient space by this equivalence relation, making no distinction between a function and its equivalence class, for common practice.

**Definition 3.34** (The space $L^p$)**.** For $1 \leq p < \infty$, we define $L^p(I, \mathbb{B})$ as the vector space of (equivalence classes) of strongly measurable functions $f : I \to \mathbb{B}$ such that

$$\int_I |f(t)|^p_{\mathbb{B}} dt < \infty.$$

We also define $L^\infty(I, \mathbb{B})$ as the vector space of bounded strongly measurable functions $f : I \to \mathbb{B}$, in the sense there exists a constant $C > 0$ such that

$$f(t) \leq C, \text{ for a.e. } t \in I$$

Endowed with the norms

$$|f|_{L^p} = \left(\int_I |f|^p_{\mathbb{B}} dt\right)^{1/p}$$

and

$$|f|_{L^\infty} = \inf\{C > 0, \ |f(t)|_{\mathbb{B}} \leq C \text{ for a.e. } t \in I\}$$

the spaces $L^p(I, \mathbb{B})$ and $L^\infty(I, \mathbb{B})$ are Banach spaces [52, 88].

#### 3.2.1.1 Mappings between Lebesgue spaces

We end this section by adapting a result from [41, Proposition 2.3], and we prove differentiability of certain mappings that are lifts on the space of curves of differentiable maps:

**Proposition 3.35** (Mappings between Lebesgue spaces)**.** *Let $U \subset \mathbb{B}$ be an open subset of a Banach space, and let $E$ and $F$ be Banach spaces. Let also $f : U \times E \to F$ be a $C^{k+1}$ map, such that for all $x \in U$, the map $f(x, \cdot) : E \to F$ is linear. Then the induced map*

$$\tilde{f} : \begin{cases} C(I,U) \times L^p(I,E) & \longrightarrow & L^p(I,F) \\ (\eta, u) & \longmapsto & f \circ (\eta, u) \end{cases}$$

*is $C^k$*

*Proof.* The case $k = 0$, i.e. if $f$ is $C^1$, directly follows from [41, Proposition 2.3]. We suppose $f$ is $C^2$. By [41, Proposition 2.3], the map $\tilde{f} : C(I,U) \times L^p(I,E) \to L^p(I,F)$ is then Gâteaux-$C^1$. Its differential is given by

$$\widetilde{df} : \begin{array}{ccc} C(I,U) \times L^p(I,E) \times C(I,\mathbb{B}) \times L^p(I,E) & \longrightarrow & L^p(I,F) \\ (\eta, u, \delta\eta, \delta u) & \longmapsto & df(\eta, u; \delta\eta, \delta u) \end{array}$$

and we directly see, since $f$ is $C^2$, that it is continuous and linear in the second variable. We prove now that the induced map $\widetilde{df} : (\eta, u) \mapsto \widetilde{df}(\eta, u; \cdot)$ from $C(I,U) \times L^p(I,E)$ to $L\left(C(I,\mathbb{B}) \times L^p(I,E), L^p(I,F)\right)$ is also continuous. We recall that $df : U \times E \to L(\mathbb{B} \times E, F)$ is continuous, and for any $(\delta\eta, u) \in C(I,\mathbb{B}) \times L^p(I,E)$ we have

$$\begin{aligned} |\widetilde{df}(\eta, u; \delta\eta, \delta u) - \widetilde{df}(\eta', u'; &\delta\eta, \delta u)|^p_{L^p(I,F)} \\ &\leq \int_I |d_{(\eta(t),u(t))} f(\delta\eta(t), \delta u(t)) - d_{(\eta'(t),u'(t))} f(\delta\eta(t), \delta u)|^p_F dt \\ &\leq \int_I C^p |(\delta\eta(t), \delta u(t))|^p_F dt \leq \int_I C^p |\delta\eta|^p_{\mathbb{B},\infty} |\delta u(t)|^p_E \\ &\leq C^p |\delta\eta|^p_{\mathbb{B},\infty} |\delta u(t)|^p_{L^p(I,E)} \end{aligned}$$

and thus we get

$$|\widetilde{df}(\eta, u; \cdot) - \widetilde{df}(\eta', u'; \cdot)|^p_{L(C(I,\mathbb{B}) \times L^p(I,E), L^p(I,F))} \leq C^p$$

which proves the continuity of $\eta, u \mapsto \widetilde{df}(\eta, u; \cdot)$, and by lemma 3.5 the map $\tilde{f}$ is Fréchet-$C^1$. The rest follows from induction. $\square$

### 3.2.2 The space of absolutely continuous curves

In this part, we define the space of absolutely continuous curves with $L^p$ derivative in a Banach manifold, following the work from Glöckner [41] and we put a Banach differentiable structure on it. Let $\mathbb{B}$ be a Banach space, $p \in [1, +\infty]$ and $a < b \in \mathbb{R}$. We define the vector space $AC_{L^p}([a,b], \mathbb{B})$ of continuous curves $\eta : [a,b] \to \mathbb{B}$ such that there exists $\gamma \in L^p([a,b], \mathbb{B})$ verifying for any $t \in [a,b]$

$$\eta(t) = \eta(a) + \int_a^t \gamma(s) ds \tag{3.1}$$

This is equivalent to saying that $\eta$ is almost everywhere differentiable with $\eta' \in L^p([a,b], \mathbb{B})$. We introduce on the space $AC_{L^p}([a,b], \mathbb{B})$ the norm $|\cdot|_{AC_{L^p}}$ given by :

$$|\eta|_{AC_{L^p}} = |\eta(a)|_{\mathbb{B}} + |\eta'|_{L^p}$$

Then $(AC_{L^p}([a,b],\mathbb{B}), |\cdot|_{AC_{L^p}})$ is a Banach space and we have the continuous inclusion :

$$AC_{L^p}([a,b],\mathbb{B}) \hookrightarrow C([a,b],\mathbb{B}) \times L^p([a,b],\mathbb{B})$$

Now let $I \subset \mathbb{R}$ an interval, and let $M$ be a Banach manifold modeled on $\mathbb{B}$. We now define the space of absolutely continuous curves in $M$:

**Definition 3.36** (Absolutely continuous curves in $M$)**.** We define $AC_{L^p}(I, M)$ as the set of curves $\eta : I \to M$, such that for any local charts $(U, \varphi)$ and any $a < b$ such that $\eta([a,b]) \subset U$, the curve

$$\varphi \circ \eta : [a,b] \to \mathbb{B}$$

is in $AC_{L^p}([a,b],\mathbb{B})$.

We turn now to the definition of a smooth manifold structure on the space of absolutely continuous curves on the space $M$. In [86], the author proved that if $M$ is a Banach manifold equipped with some strong Riemannian metric, then $AC_{L^p}(I, M)$ also remains a Banach manifold. Following an older work by Krikorian [60], that was continued in [42] for regulated curves, it is however possible to put a differentiable structure on such a space, without any other hypotheses on the manifold $M$. We also refer to the recent work [83] that also defines a differentiable structure in the space of absolutely continuous curves with values in a general infinite dimensional manifold.

**Theorem 3.37** (Differential structure on $AC_{L^p}(I, M)$)**.** *Assume $I$ is a compact interval of $\mathbb{R}$. Then we get the following*

1. *The space $AC_{L^p}(I, M)$ is a Banach manifold.*
2. *For $t \in I$, the evaluation*

$$\mathrm{ev}_t : \left\{ \begin{array}{ccc} AC_{L^p}(I, M) & \longrightarrow & M \\ \eta & \longmapsto & \eta(t) \end{array} \right.$$

*is smooth.*

3. *The vector bundle $AC_{L^p}(I, TM) \to AC_{L^p}(I, M)$ can be taken as the tangent bundle. For $\eta \in AC_{L^p}(I, M)$, we therefore have*

$$T_\eta AC_{L^p}(I, M) = AC_{L^p}(I \leftarrow \eta^* TM)$$

*where $AC_{L^p}(I \leftarrow \eta^* TM) = \{\gamma \in AC_{L^p}(I, TM),\ \gamma(t) \in T_{\eta(t)}M,\ \forall t \in I\}$*

We prove this result in the next subsection

### 3.2.3 Proof of theorem 3.37

Let $\mathcal{A}$ be the set of all the $\alpha = (a^\alpha, U^\alpha, \varphi^\alpha)$ such that there exist a positive integer $n = n^\alpha$, a family $a^\alpha = (a_i^\alpha)_{0 \le i \le n}$ such that $0 = a_0 < \ldots < a_n = 1$ and two families $U^\alpha = (U_i^\alpha)_{1 \le i \le n}$ and $\varphi^\alpha = (\varphi_i^\alpha)_{1 \le i \le n}$ such that $(U_i^\alpha, \varphi_i^\alpha)$ is a chart on $M$ with $\varphi_i^\alpha : U_i^\alpha \to V_i^\alpha = \varphi_i^\alpha(U_i^\alpha) \subset \mathbb{B}$ for $1 \le i \le n$. We will denote $I^\alpha = (I_i^\alpha)_{1 \le i \le n}$ where $I_i^\alpha = [a_{i-1}^\alpha, a_i^\alpha]$.

For any $\alpha \in \mathcal{A}$ and $n = n^\alpha$, we denote $\mathcal{U}^\alpha = AC_{L^p}(a^\alpha; U^\alpha)$ where

$$AC_{L^p}(a^\alpha; U^\alpha) = \{\, \eta \in C([0,1], M) \mid \eta(I_i^\alpha) \subset U_i^\alpha \text{ and } \varphi_i^\alpha \circ \eta_{|I_i^\alpha} \in AC_{L^p}(I_i^\alpha, \mathbb{B}),\ 1 \leq i \leq n \,\},$$

$E^\alpha = \prod_{i=1}^n AC_{L^p}(I_i^\alpha; \mathbb{B})$ and $\Phi^\alpha : \mathcal{U}^\alpha \to E^\alpha$ the one-to-one mapping defined by

$$\Phi^\alpha(\eta) = (\varphi_i^\alpha \circ \eta_{|I_i^\alpha})_{1 \leq i \leq n}\,.$$

We denote $\mathcal{N}^\alpha = \Phi^\alpha(\mathcal{U}^\alpha)$.

**Proposition 3.38.** *We can create a differentiable structure on $AC_{L^p}([0,1], M)$ by showing the two following facts:*

1. *For any $\alpha \in \mathcal{A}$, $\mathcal{N}^\alpha$ is a submanifold of $E^\alpha$.*
2. *For any $\alpha, \beta \in \mathcal{A}$ such that $\mathcal{U}^{\alpha,\beta} = \mathcal{U}^\alpha \cap \mathcal{U}^\beta \neq \emptyset$, then $\Phi^\alpha(\mathcal{U}^{\alpha,\beta})$ (resp. $\Phi^\beta(\mathcal{U}^{\alpha,\beta})$) is an open subset of $\mathcal{N}^\alpha$ (resp. $\mathcal{N}^\beta$) and $\Phi^{\alpha,\beta} = \Phi^\beta \circ (\Phi^\alpha)^{-1} : \Phi^\alpha(\mathcal{U}^{\alpha,\beta}) \to \Phi^\beta(\mathcal{U}^{\alpha,\beta})$ is a smooth diffeomorphism.*

*A smooth atlas is then given by the family of charts $(\mathcal{U}_\psi^\alpha, \Phi_\psi^\alpha)$, for any $\alpha \in \mathcal{A}$ and any chart $(\mathcal{V}_\psi, \psi)$ on $\mathcal{N}^\alpha$ where $\mathcal{U}_\psi^\alpha = (\Phi^\alpha)^{-1}(\mathcal{V}_\psi)$, and $\Phi_\psi^\alpha = \psi \circ \Phi^\alpha$.*

**Remark 3.39.** *In this construction, the family of sets $(\mathcal{U}_\psi^\alpha)$ defines the manifold topology of $AC_{L^p}(I, M)$, and the sets $\mathcal{U}^\alpha$ are open submanifolds of $AC_{L^p}(I, M)$.*

*Proof.* We first show 1: For $n = 1$, the result is trivial. For $n > 1$, if we introduce the open set $\mathcal{V}^\alpha = \{\, \eta \in \prod_{i=1}^n AC_{L^p}(I_i^\alpha; V_i^\alpha) \mid \eta_i(a_i) \in \varphi_i^\alpha(U_{i+1}^\alpha)\ \forall 1 \leq i \leq n-1\}$ of $E^\alpha$, we have $\mathcal{N}^\alpha \subset \mathcal{V}^\alpha$. Moreover, we have $\mathcal{N}^\alpha = (\sigma^\alpha)^{-1}(0)$ for $\sigma^\alpha : \mathcal{V}^\alpha \to \prod_{i=2}^n \mathbb{B}$ such that $\sigma^\alpha(\eta) = (\varphi_{i+1}^\alpha \circ (\varphi_i^\alpha)^{-1} \circ \eta_i(a_i^\alpha) - \eta_{i+1}(a_i^\alpha))_{1 \leq i < n}$. Since $\sigma^\alpha$ is smooth, it is enough to conclude to show that at any point $\eta \in \mathcal{N}^\alpha$, $\sigma^\alpha$ is a submersion at $\eta$. However, considering the closed subspace $H_n = \{\, (0, \delta\eta_2, \cdots, \delta\eta_n) \mid \delta\eta_i : I_i^\alpha \to \mathbb{B} \text{ is contant} \,\}$ of $T_\eta \mathcal{V}^\alpha \simeq \prod_{i=1}^n AC_{L^p}(I_i^\alpha, \mathbb{B})$, we get immediatly that $T_\eta \sigma^\alpha$ is surjective and $\operatorname{Ker} T_\eta \sigma^\alpha \oplus H_n = T_\eta \mathcal{V}^\alpha$ so that its kernel splits (see [61] prop 2.2)

$$\begin{tikzcd}
& \mathcal{U}^{\alpha,\beta} = \mathcal{U}^\alpha \cap \mathcal{U}^\beta \arrow[dl, "\Phi^\alpha"'] \arrow[dr, "\Phi^\beta"] & \\
\Phi^\alpha(\mathcal{U}^{\alpha,\beta}) \arrow[rr, "\Phi^{\alpha,\beta}"] \arrow[d, "j^\alpha"] & & \Phi^\beta(\mathcal{U}^{\alpha,\beta}) \arrow[d, "j^\beta"'] \\
\mathcal{V}^{\alpha,\beta} \arrow[rr, "\Psi^{\alpha,\beta}"] & & \mathcal{V}^{\beta,\alpha}
\end{tikzcd}$$

Let us prove 2: Let first remark that there exists a unique $\gamma = (a^\gamma, U^\gamma)$ such that $\operatorname{range}(a^\gamma) = \operatorname{range}(a^\alpha) \cup \operatorname{range}(a^\beta)$, $\mathcal{U}^{\alpha,\beta} = \mathcal{U}^\gamma$ with $U_k^\gamma = U_{i_k}^\alpha \cap U_{j_k}^\beta$ for $(i_k, j_k)$ the unique pair $(i,j)$ for which $I_k^\gamma = I_i^\alpha \cap I_j^\beta$. We check that $\Phi^\alpha(\mathcal{U}^{\alpha,\beta}) = \mathcal{N}^\alpha \cap \prod_{i=1}^{n^\alpha} \mathcal{V}_i^\alpha$ with $\mathcal{V}_i^\alpha = \{\, \eta_i \in AC_{L^p}(I_i^\alpha, \mathbb{B}) \mid \eta_i(I_k^\gamma) \subset \phi_i^\alpha(U_k^\gamma),\ k \text{ s.t. } i = i_k \,\}$ open in $AC_{L^p}(I_i^\alpha, \mathbb{B})$ so that $\Phi(\mathcal{U}^{\alpha,\beta})$ is an open set of $\mathcal{N}^\alpha$.

We consider the continuous linear injectif mapping

$$j^\alpha : E^\alpha \to E^{\alpha,\beta} = \prod_{1\le k\le n^\gamma} AC_{L^p}(I_k^\gamma, \mathbb{B})$$

defined by $j^\alpha(\eta) = (\eta_{i_k|I_k^\gamma})_{1\le k\le n^\gamma}$. We check that $j^\alpha$ is an isomorphism onto its image since $E^{\alpha,\beta} = j^\alpha(E^\alpha) \oplus \prod_{k=1}^{n^\gamma}\{\ \delta\eta_k \in AC_{L^p}(I_k^\gamma, \mathbb{B}) \mid \delta\eta_k \equiv c\mathbf{1}_{i_k=0} \text{ for } c \in \mathbb{R}\ \}$ so that $j^\alpha(E^\alpha)$ splits (see [61] prop 2.2). Similarly $j^\beta : E^\beta \to E^{\beta,\alpha}$ is an isomorphism onto its image. Moreover, we have a smooth diffeomorphic mapping $\Psi_{\alpha,\beta} : \mathcal{V}^{\alpha,\beta} \to \mathcal{V}^{\beta,\alpha}$ between the open set $\mathcal{V}^{\alpha,\beta} = \prod_{k=1}^{n^\gamma} AC_{L^p}(I_k^\gamma, \varphi_{i_k}^\alpha(U_k^\gamma)) \subset E^{\alpha,\beta}$ and the open set $\mathcal{V}^{\beta,\alpha} = \prod_{k=1}^{n^\gamma} AC_{L^p}(I_k^\gamma, \varphi_{j_k}^\beta(U_k^\gamma)) \subset E^{\beta,\alpha}$ given by $\Psi^{\alpha,\beta}(\eta = (\eta_k)_{1\le k\le n^\gamma}) = (\varphi_{j_k}^\beta \circ (\varphi_{i_k}^\alpha)^{-1} \circ \eta_k)_{1\le k\le n^\gamma}$. Since $j^\alpha(\Phi^\alpha(\mathcal{U}^{\alpha,\beta})) \subset \mathcal{V}^{\alpha,\beta}$ and for any $\eta \in \mathcal{U}^{\alpha,\beta}$ we have $\Psi^{\alpha,\beta} \circ j^\alpha \circ \Phi^\alpha(\eta) = j^\beta \circ \Phi^\beta(\eta)$ we get that $\Phi^{\alpha,\beta} = (j^\beta)^{-1} \circ \Psi^{\alpha,\beta} \circ j^\alpha_{|\Phi^\alpha(\mathcal{U}^{\alpha,\beta})}$ so that $\Phi^{\alpha,\beta}$ is a smooth diffeomorphism. □

We now want to characterize the tangent bundle of $AC_{L^p}(I, M)$. We first start by proving the following result

**Proposition 3.40.** *The evalutation map* $\mathrm{ev}_{t_0} : AC_{L^p}(I, M) \to M$, *where* $t_0 \in I$ *is smooth.*

*Proof.* We prove this result using the previous construction. Let $t_0 \in I$, let $\eta \in AC_{L^p}(I, \mathcal{M})$, and let $\mathcal{U}^\alpha = AC_{L^p}(a^\alpha; U^\alpha)$ such that $\eta$ is in $\mathcal{U}^\alpha$. Let also $i \in \{1, .., n\}$ such that $t_0 \in I_i^\alpha$, and thus

$$\mathrm{ev}_{t_0}(\mathcal{U}^\alpha) \subset U_i^\alpha$$

It is equivalent to prove the smoothness of the induced evaluation mapping $\tilde{\mathrm{ev}}_{t_0} = \varphi_i^\alpha \circ \mathrm{ev}_{t_0} \circ (\Phi^\alpha)^{-1} : \mathcal{N}^\alpha \to V_i^\alpha$, and given by :

$$\tilde{\mathrm{ev}}_{t_0} : \left\{ \begin{array}{ccc} \mathcal{N}^\alpha & \longrightarrow & V_i^\alpha \\ (\eta_1, \dots, \eta_n) & \longmapsto & \eta_i(t_0) \end{array} \right.$$

This mapping is the restriction of a linear mapping $\tilde{\mathrm{ev}}_{t_0} : E^\alpha \to \mathbb{B}$, which is continuous since the evaluation is continuous in $AC_{L^p}(I_i^\alpha, \mathbb{B})$ [41], and thus $\tilde{\mathrm{ev}}_{t_0} : \mathcal{N}^\alpha \to V_i^\alpha$ is smooth. □

Now we can finally characterize the tangent bundle of $AC_{L^p}(I, M)$, and we prove it is isomorphic to the bundle $AC_{L^p}(I, TM) \to AC_{L^p}(I, M)$

**Proposition 3.41.** *Denote by*

$$\pi_{TAC} : TAC_{L^p}(I, M) \to AC_{L^p}(I, M)$$

*the tangent bundle of* $AC_{L^p}(I, M)$. *Let* $\pi_* : AC_{L^p}(I, TM) \to AC_{L^p}(I, M)$ *given by*

$$\pi_* : \left\{ \begin{array}{ccc} AC_{L^p}(I, TM) & \longrightarrow & AC_{L^p}(I, M) \\ \gamma & \longmapsto & \pi \circ \gamma \end{array} \right.$$

*where* $\pi = \pi_{TM} : TM \to M$ *is the tangent bundle of* $M$. *Then* $\pi_* : AC_{L^p}(I, TM) \to AC_{L^p}(I, M)$ *can be taken as the tangent bundle of* $AC_{L^p}(I, M)$ *through the isomorphism*

$$\zeta : \left\{ \begin{array}{ccc} TAC_{L^p}(I, M) & \longrightarrow & AC_{L^p}(I, TM) \\ w & \longmapsto & [t \mapsto T_{\pi_{TAC}(w)} \mathrm{ev}_t\, w] \end{array} \right. \tag{3.2}$$

*Proof.* Since the evaluation mapping is smooth, the curve $\gamma : t \mapsto T_{\pi_{TAC}(w)} \operatorname{ev}_t w$ is well defined in $TM$. Let us first show that it is an absolutely continuous curve. We denote $\eta = \pi_{TAC}(w)$, and we have $\eta(t) = \operatorname{ev}_t(\eta) = \pi(\gamma(t))$. Like previously in proposition 3.38, we can define an open submanifold $(\mathcal{U}^{\alpha,M}, \Phi^{\alpha,M})$ of $AC_{L^p}(I, M)$, where $\mathcal{U}^{\alpha,M} = AC_{L^p}(a^\alpha, U^{\alpha,M})$ for some open charts $(U_i^{\alpha,M}, \varphi_i^{\alpha,M})$ of $M$. We suppose $\eta \in \mathcal{U}^{\alpha,M}$. We can now also define an induced submanifold $(\mathcal{U}^{\alpha,TM}, \Phi^{\alpha,TM})$ of $AC_{L^p}(I, TM)$, where $\mathcal{U}^{\alpha,TM} = AC_{L^p}(a^\alpha, U^{\alpha,TM})$, $U_i^{\alpha,TM} = TU_i^{\alpha,M}$, and $\varphi_i^{\alpha,TM} = T\varphi_i^{\alpha,M}$. The diffeomorphism $\Phi^{\alpha,TM} : \mathcal{U}^{\alpha,TM} \to \mathcal{N}^{\alpha,TM}$ is defined as previously with the diffeomorphisms $\varphi_i^{\alpha,TM}$. This gives us the following commutative diagram

$$\begin{tikzcd}
\mathcal{U}^{\alpha,M} \arrow[d, "\Phi^{\alpha,M}"] & \mathcal{U}^{\alpha,TM} \arrow[l, "\pi_*"'] \arrow[d, "\Phi^{\alpha,TM}"] \\
\mathcal{N}^{\alpha,M} & \mathcal{N}^{\alpha,TM}
\end{tikzcd}$$

Since $\eta \in \mathcal{U}^{\alpha,M}$, and $\eta(t) = \pi(\gamma(t))$, therefore for all $1 \le i \le n$, $\gamma(I_i^\alpha) \subset U_i^{\alpha,TM} = TU_i^{\alpha,M}$, such that $T\varphi_i^\alpha \circ \gamma_{|I_i^\alpha}$ is in $TV_i^\alpha$ and for $t \in I_i^\alpha$

$$\begin{aligned}
T\varphi_i^\alpha \circ \gamma_{|I_i^\alpha}(t) &= T(\varphi_i^\alpha \circ \operatorname{ev}_t)w \\
&= T(\varphi_i^\alpha \circ \operatorname{ev}_t \circ (\Phi^{\alpha,M})^{-1}) T\Phi^{\alpha,M} w
\end{aligned}$$

Since $\tilde{w} = T\Phi^{\alpha,M} w \in T\mathcal{N}^\alpha \subset \prod_{i=1}^n AC_{L^p}(I_i^\alpha, V_i^\alpha) \times AC_{L^p}(I_i^\alpha, \mathbb{B})$, we thus have $T\varphi_i^\alpha \circ \gamma_{|I_i^\alpha}(t) = \tilde{w}(t)$, and then $\gamma_{|I_i^\alpha} \in AC_{L^p}(I_i^\alpha, U_i^{\alpha,TM})$ and $\gamma$ is continuous. Therefore $\gamma$ is in $\mathcal{U}^{\alpha,TM} \subset AC_{L^p}(I, TM)$, and $\zeta(T\mathcal{U}^{\alpha,M}) \subset \mathcal{U}^{\alpha,TM}$.

Let us prove now $\zeta : T\mathcal{U}^{\alpha,M} \to \mathcal{U}^{\alpha,TM}$ is smooth. From the above, we obtain the following diagram

$$\begin{tikzcd}
T\mathcal{U}^{\alpha,M} \arrow[r, "\zeta"] \arrow[d, "T\Phi^{\alpha,M}"] & \mathcal{U}^{\alpha,TM} \arrow[d, "\Phi^{\alpha,TM}"] \\
T\mathcal{N}^{\alpha,M} \arrow[d, hook] & \mathcal{N}^{\alpha,TM} \arrow[d, hook] \\
TE^\alpha \arrow[r, "\sim"] & \prod_{i=1}^n AC_{L^p}(I_i^\alpha, \mathbb{B} \times \mathbb{B})
\end{tikzcd}$$

And we can verify that under the identification $TE^\alpha \simeq \prod_{i=1}^n AC_{L^p}(I_i^\alpha, \mathbb{B}\times\mathbb{B})$, we also have the identification $T\mathcal{N}^{\alpha,M} \simeq \mathcal{N}^{\alpha,TM}$, and this identification is exactly given by $\Phi^{\alpha,TM} \circ \zeta \circ (T\Phi^{\alpha,M})^{-1}$. Therefore $\zeta : T\mathcal{U}^{\alpha,M} \to \mathcal{U}^{\alpha,TM}$ is a smooth bundle isomorphism with base space $\mathcal{U}^{\alpha,M}$ □

### 3.2.4 Ordinary differential equations of Carathéodory type

We finish this section by recalling some results on ordinary differential equations in Banach spaces, and in particular when the regularity in time is only $L^p$. Indeed, we will often have to integrate equations with $L^2$ regularity in time in the next sections, particularly when considering sub-Riemannian metrics in groups. The theory for such ODE in Banach spaces can be found in [27, 98], and we adapt it to the setting of absolutely continuous curves as in [41]. Let $I$ be an open interval of $\mathbb{R}$, $\mathbb{B}$ a Banach space, $U \subset \mathbb{B}$ an open subset and $y_0 \in U$. We consider a map $f : I \times U \to \mathbb{B}$, and we are interested in the equation

$$\begin{cases} \dot{y}(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases} \tag{3.3}$$

**Definition 3.42** (Carathéodory property)**.** We say that $f$ has the *Carathéodory property* if

1. For every $t \in I$, the map $f(t, .) : U \to \mathbb{B}$ is continuous,
2. For every $x \in U$, the map $f(., x) : I \to \mathbb{B}$ is strongly measurable.

We can now state the main result on this ODE, namely the Picard-Lindelof theorem that gives existence and unicity of solutions.

**Definition 3.43** (Carathéodory solutions)**.** We say that $y : I \to U$ is a *Carathéodory solution* of (3.3) if $y \in AC_{L^1}(I, \mathbb{B})$, $y(I) \subset U$, $y(0) = y_0$ and $\dot{y}(t) = f(t, y(t))$ for a.e. $t \in I$.

We denote by $\mathrm{Lip}(\mathbb{B}, \mathbb{B})$ the space of Lipschitz functions of $\mathbb{B}$. For any $v \in \mathrm{Lip}(\mathbb{B}, \mathbb{B})$, we also define $\mathrm{Lip}(v)$ its Lipschitz constant by

$$\mathrm{Lip}(v) = \inf \left\{ \frac{|f(x) - f(y)|_{\mathbb{B}}}{|x - y|_{\mathbb{B}}} \mid x \neq y \in \mathbb{B} \right\}.$$

We recall that, endowed with the norm

$$|v|_{\mathrm{Lip}} = |v(x^*)|_{\mathbb{B}} + \mathrm{Lip}(v),$$

where $x^* \in \mathbb{B}$ is a fixed point, the space $\mathrm{Lip}(\mathbb{B}, \mathbb{B})$ is a Banach space. The next theorem can be found in [98, Theorem C.6]

**Theorem 3.44** (Picard-Lindelof theorem : Carathéodory version)**.** *Suppose $f : I \times U \to \mathbb{B}$ is of Carathéodory type, and that, seen as a map from $I$ to $C(U, \mathbb{B})$, $f$ is in $L^1(I, \mathrm{Lip}(U, \mathbb{B}))$. Then there exists a unique solution $y \in AC_{L^1}(I, \mathbb{B})$ to the equation* (3.3)*.*

## 3.3 Variational problems in Banach spaces and Pontryagin principle

We finish this chapter with an application to the minimization of a class of variational problems. We will use particular tools from optimal control to characterize minimizers. The variational problems developed here will serve as models for many applications of shape analysis and registration developped in this thesis.

Let $\mathcal{Q}$ be a Banach manifold, modeled on a Banach space $\mathbb{B}$, and $V$ a Banach space, and denote $I = [0, 1]$. Let $q_0 \in \mathcal{Q}$. In this section, we are interested in the control system

$$\dot{q}(t) = f(q(t), u(t)), \; q(0) = q_0 \tag{3.4}$$

where $u \in L^1(I, V)$, and $f : \mathcal{Q} \times V \to T\mathcal{Q}$ is $C^2$, and is a fiber bundle morphism over the identity of $\mathcal{Q}$ (this means that for all $(q, v) \in \mathcal{Q} \times V$, then $f(q, v)$ is in $T_q\mathcal{Q}$). We denote by $\mathcal{U}_{q_0}$ the set of admissible controls $u \in L^1(I, V)$ such that it leads to a maximal trajectory $q : [0, T] \to \mathbb{R}$ starting in $q(0) = q$ and such that $T > 1$, and we suppose that the set $\mathcal{U}_{q_0}$ is open in $L^1(I, V)$ (see for instance [7, 98, 4]).

Let $\mathcal{D} : \mathcal{Q} \to \mathbb{R}^+$ be a $C^1$ function and

$$L : \mathcal{Q} \times V \to \mathbb{R}$$

be a $C^1$ Lagrangian. We are interested in this section on functionals $J : \mathcal{U}_{q_0} \to \mathbb{R}$ that take the form :

$$J(u) = \int_I L(q(t), u(t))dt + \mathcal{D}(q(1))$$

with $u \in \mathcal{U}_{q_0}$, and such that $q \in AC_{L^1}([0, 1], \mathcal{Q})$ satisfies the dynamic 3.4. Here the Lagrangian $L$ gives the energy along the curve $t \mapsto (q(t), u(t))$ and the map $\mathcal{D}$ gives an endpoint constraint. We suppose that, for any charts $U \subset \mathcal{Q}$, there exists continuous functions $\gamma_0 : \mathbb{R} \to \mathbb{R}$ and $\gamma_1 : U \to \mathbb{R}$ such that $\gamma_0(0) = 0$ and

$$\|dL_{q,u} - dL_{q',u'}\| \leq \gamma_0(|q - q'|_{\mathbb{B}}) + \gamma_1(q - q')|u - u'|_V \tag{3.5}$$

for any $q, q' \in U$ and $u, u' \in V$. This condition 3.5 ensures that the mapping $J_0 : AC_{L^1}(I, \mathcal{Q}) \times L^1(I, V) \to \mathbb{R}$ defined by $J_0(q, u) = \int_I L(q(t), u(t))dt$ is well defined and continuously differentiable [8] and that

$$dJ_0(q, u)(\delta q, \delta u) = \int_I \frac{\partial}{\partial q} L(q(t), u(t))\delta q(t) + \frac{\partial}{\partial u} L(q(t), u(t))\delta u(t)dt \tag{3.6}$$

In this section, we want to minimize the functional $J$ and characterize its critical points, which is a problem of optimal control theory in Banach spaces. We start by proving the evolution map has some regularities

**Lemma 3.45** (Evolution map associated with the dynamic)**.** *The evolution map*

$$\mathrm{Evol}_{q_0} : u \in \mathcal{U}_{q_0} \mapsto q^u \in AC_{L^1}(I, \mathcal{Q})$$

*where $q^u$ is solution of equation* (3.4)*, is $C^1$. For $u \in \mathcal{U}_{q_0}, \delta u \in L^1(I, V)$, its derivative $\delta q = T_u \mathrm{Evol}_{q_0} \cdot \delta u$ is solution of the linear Cauchy problem*

$$\delta\dot{q}(t) = \partial_q \left(f(q, u(t))\right)_{|q=q^u(t)} \delta q(t) + \partial_u (f(q^u(t), u)_{|u=u(t)} \delta u(t), \;\; \delta q(0) = 0 \tag{3.7}$$

*Proof.* The proof is quite similar to the proof of proposition 4.22 (where the calculus in the charts will be more detailed). Let $u^* \in \mathcal{U}_{q_0}$, and $q^u = \mathrm{Evol}_{q_0}(u)$. Similarly to proposition 4.22, we will consider a chart $\mathcal{V} \subset AC_{L^1}(I, \mathcal{Q})$ around $q^{u*}$, and we will even actually reduce to the case $\mathcal{V} = AC_{L^1}(I, \mathbb{B})$. We consider the mapping

$$C : \left\{ \begin{array}{ccc} AC_{L^1}(I, \mathbb{B}) \times \mathcal{U}_{q_0} & \longrightarrow & \mathbb{B} \times L^1(I, \mathbb{B}) \\ (q, u) & \longmapsto & (q(0), \dot{q} - f(q, u)) \end{array} \right.$$

The mapping $C$ is $C^1$ since $f$ is $C^2$ and by [41, prop 2.3]. Moreover, the derivative with regards to the first variable is given by

$$\partial_q C(q, u)\delta q = (\delta q(0), \delta\dot{q} - \partial_q f(q, u)\delta q)$$

and is a Banach isomorphism (by Banach-Schauder theorem). By implicit function theorem, and since $C(\mathrm{Evol}(u), u) = (q_0, 0)$, there exists an open neighborhood $W \subset L^1(I, \mathbb{B})$ of $u^*$ such Evol is $C^1$, and the curve $\delta q = T_u \,\mathrm{Evol}\, \delta u$ is solution of the linear cauchy problem 3.7. □

We also denote $\mathrm{End}_{\mathcal{Q}} : \mathcal{U}_{q_0} \to \mathcal{Q}$ the endpoint mapping, defined by

$$\mathrm{End}_{q_0}(u) = \mathrm{Evol}_{q_0}(u)(1)$$

We study now how we can minimize the functional $J$, and we characterize its critical points. Let recall that the Banach manifold $T^*\mathcal{Q}$ can be equipped with a canonical (weak) symplectic form $\omega$ (that we define introducing the Liouville form and its exterior derivative [9]). In local charts of $T^*\mathcal{Q}$, the closed form $\omega$ is defined by

$$\omega_{q,p}\left((\delta q, \delta p), (\delta q', \delta p')\right) = (\delta p \,|\, \delta q') - (\delta p' \,|\, \delta q)$$

Let $\mathcal{H} : T^*\mathcal{Q} \times V \to \mathbb{R}$ be the pre-Hamiltonian associated with the function $J$ defined by

$$\mathcal{H}(q, p, u) = (p \,|\, f(q, u)) - L(q, u)$$

Since $f$ is $C^2$, and the duality pairing $((q, p), (q, X)) \in TG^* \oplus_Q TG \mapsto (p \,|\, X) \in \mathbb{R}$ is smooth, the Hamiltonian $\mathcal{H}_{\mathcal{Q}} : T^*\mathcal{Q} \times V \longrightarrow \mathbb{R}$ is thus $C^2$. In local coordinates, the partial derivative $\partial_p \mathcal{H}_{\mathcal{Q}}(q, p, u)$ is given by:

$$\forall \delta p \in T_q^*\mathcal{Q}, \ \partial_p \mathcal{H}_{\mathcal{Q}}(q, p, u)\delta p = (\delta p \,|\, f(q, u))$$

so that

$$\partial_p \mathcal{H}_{\mathcal{Q}}(q, p, u) \simeq f(q, u) \in T_q\mathcal{Q}.$$

Therefore there exists a symplectic gradient $\nabla^\omega \mathcal{H}_{\mathcal{Q}}(q, p, u) \in T_{q,p}T^*\mathcal{Q}$ for every $u \in V$. In canonical charts of $T^*\mathcal{Q}$, we have :

$$\nabla^\omega \mathcal{H}_{\mathcal{Q}}(q, p, u) = \left(\partial_p \mathcal{H}_{\mathcal{Q}}(q, p, u), \ -\partial_q \mathcal{H}_{\mathcal{Q}}(q, p, u)\right)$$

We get the following result :

**Theorem 3.46** (Critical points of $J$)**.** *Let $u^* \in \mathcal{U}_{q_0}$ be a critical point of $J$, that is $dJ(u^*) = 0$, and denote by $q = \mathrm{Evol}(u) \in AC_{L^1}(I, Q)$ the solution of equation 3.4. Then there exists a costate $t \mapsto p(t) \in T^*_{q(t)}\mathcal{Q}$ in $AC_{L^1}(I, T^*\mathcal{Q})$ such that $(q, p, u)$ satisfies the Hamiltonian equations :*

$$\begin{cases} (\dot{q}, \ \dot{p}) = \nabla^\omega \mathcal{H}_{\mathcal{Q}}(q, p, u) \\ \partial_u \mathcal{H}_{\mathcal{Q}}(q, p, u) = 0 \end{cases} \tag{3.8}$$

*Conversely, if there exists such a costate $p$ such that $(q, p, u^*)$ satisfies the Hamiltonian equations 3.8, then $u^*$ is a critical point of $J$.*

*Proof.* The proof is similar to the one given in [7, 8]. Let $u \in \mathcal{U}_{q_0}$, and $q = \mathrm{Evol}(u)$. We introduce the costate $p_1 = -d\mathcal{D}(q_1) \in T^*_{q_1}\mathcal{Q}$ and we define $p \in AC_{L^1}(I, T^*\mathcal{Q})$ as the solution of the following linear Cauchy problem :

$$\begin{cases} \dot{p}(t) = -\partial_q \mathcal{H}_{\mathcal{Q}}(q(t), p(t), u(t)) = -\left(\partial_q (f(q, u(t))_{q=q(t)}\right)^* p(t) + \partial_q (L(q, u(t))_{q=q(t)} \\ p(1) = p_1 \end{cases} \tag{3.9}$$

We now compute the differential of the mapping $J$ and we prove that for all $\delta u \in L^2(I, V)$,

$$dJ(u)\delta u = -\int_I \partial_u \mathcal{H}(q(t), p(t), u(t))\delta u dt$$

Let $\delta u \in L^2(I, V)$, and $\delta q = T_u \operatorname{Evol}(\delta u)$. Recall that

$$J(u) = J_0(\operatorname{Evol}(u), u) + \mathcal{D}(\operatorname{Evol}(u)(1)),$$

so that we have

$$\begin{aligned} dJ(u)\delta u &= dJ_0(q, u)(\delta q, \delta u) + d\mathcal{D}(q_1)(\delta q) \\ &= \int_I \frac{\partial}{\partial q} L(q(t), u(t))\delta q(t) + \frac{\partial}{\partial u} L(q(t), u(t))\delta u(t) dt - (p_1 \,|\, \delta q(1)) \end{aligned}$$

Thus the curve $\delta q : I \to TQ$ satisfies the linear Cauchy problem 3.7

$$\delta q(0) = 0, \;\; \delta \dot{q}(t) = \partial_q \left( f(q, u(t))_{|q=q(t)} \, \delta q(t) + \partial_u (f(q(t), u)_{|u=u(t)} \delta u(t). \right.$$

Now, since $t \mapsto p(t)$ is also solution of the linear Cauchy equation 3.9 and using integration by part, we find that

$$\begin{aligned} (p(1) \,|\, \delta q(1)) &= (p(0) \,|\, \delta q(0)) + \int_I (\dot{p}(t) \,|\, \delta q(t)) + (p(t) \,|\, \delta \dot{q}(t)) \, dt \\ &= \int_I -\Big( (\partial_q \left( f(q, u(t))_{|q=q(t)} \right)^* p(t) - \partial_q (L(q, u(t)))_{q=q(t)} \,|\, \delta q(t) \Big) \\ &+ \Big( p(t) \,|\, \partial_q \left( f(q, u(t))_{|q=q(t)} \, \delta q(t) + \partial_u (f(q(t), u)_{|u=u(t)} \delta u(t) \right. \Big) dt \\ &= \int_I \big( p(t) \,|\, \partial_u (f(q(t), u)_{|u=u(t)} \delta u(t) \big) + \partial_q (L(q, u(t))_{|q=q(t)} \delta q(t) dt \end{aligned}$$

Combining this with the derivative $dJ_0(q, u)$, we finally get

$$\begin{aligned} dJ(u)\delta u &= \int_I \partial_u \left( L(q(t), u(t)) \right) \delta u(t) - (p(t) \,|\, \partial_u \left( f(q(t), u(t)) \right) \delta u(t)) \, dt \\ &= -\int_I \partial_u \mathcal{H}(q(t), p(t), u(t))\delta u dt \end{aligned}$$

Therefore we have the following equivalence :

$$dJ(u) = 0 \iff \forall t, \; \partial_u \mathcal{H}(q(t), p(t), u(t)) = 0 \tag{3.10}$$

which concludes the proof. □

**Remark 3.47.** *In the case the pre-Hamiltonian is concave in $u$, the condition*

$$\partial_u \mathcal{H}(q(t), p(t), u^*(t)) = 0$$

*is equivalent to the condition $\mathcal{H}(q(t), p(t), u^*(t)) = \max_{u \in V} \mathcal{H}(q(t), p(t), u)$. This leads us to define the Hamiltonian*

$$h(q, p) = \max_{u \in V} \mathcal{H}(q, p, u).$$

*Moreover if there exists a $C^1$ mapping $u : T^*\mathcal{Q} \to V$ such that*

$$h(q,p) = \mathcal{H}(q,p,u(q,p)),$$

*then $h$ also admits a symplectic gradient and*

$$\nabla^\omega h(q,p) = \nabla^\omega \mathcal{H}(q,p,u(q,p))$$

*where $\nabla^\omega \mathcal{H}(q,p,u(q,p))$ denotes the partial symplectic gradient with regards to the first two variables [7]*

# Chapter 4

# Mathematical background II : Half-Lie groups

## Contents



In this chapter, we study groups endowed with a Banach differential structure that is, in some sense, compatible with the group operation. The main model example, which is the most common in the LDDMM framework, is the group of diffeomorphisms of $\mathbb{R}^d$ or of a compact manifold $M$ with finite regularity ($C^k$, Sobolev, Hölder, etc). These diffeomorphism groups are topological groups that can be equipped with a Banach differential structure, where right translations are smooth. However the left translations are only continuous, due to a loss of regularity observed, which implies that these groups are not Lie groups. In particular most of the tools of Lie group theory cannot be applied.

More generally, in order to keep a Banach structure, we are forced [78, 79] to deal with topological groups such that only the right translations are smooth. Such objects are referred to as half-Lie groups [63, 16]. In most examples, the left translations and the inverse map may fail to be smooth.

These groups and several of their properties were recently studied [63] and later in [16], where the authors recovered many fundamental results. We begin by following their exposition and review some general aspects of these groups in section 4.2, recalling in particular the definitions of $C^k$-differentiable elements, which allow to regain some regularity for the left translations.

In section 4.3, we study $L^p$ regularity of the half-Lie groups, that is we study $L^p$ curves in the tangent space of the identity and whether they can be integrated in the half-Lie group. In particular, we generalize the result of [16, theorem 4.2], which establishes smooth regularity for the groups of $C^k$-differentiable elements, where $k \geq 1$. Related studies of $L^p$ regularity on half-Lie groups were also recently carried out in [83].

Finally, section 4.5 is devoted to the definition of weak Poisson structures on half-Lie groups, and in particular to an analoguous version of the Lie-Poisson reduction as developed in [65].

## 4.1 The example of $\operatorname{Diff}_{C_0^k}(\mathbb{R}^d)$

Let $k \geq 1$. We introduce here the group of $C^k$ diffeomorphisms as one of the main example of Banach half-Lie groups. In the LDDMM framework, the most common example in applications is the group of $C^k$ diffeomorphisms that tend to the identity at infinity, and whose derivatives also tend to the identity at infinity:

$$\operatorname{Diff}_{C_0^k}(\mathbb{R}^d) = \left(\operatorname{id} + C_0^k(\mathbb{R}^d, \mathbb{R}^d)\right) \cap \operatorname{Diff}^1(\mathbb{R}^d).$$

It is used as the group of deformations to transport images, or general shapes. The group $\operatorname{Diff}_{C_0^k}(\mathbb{R}^d)$ is open [70] in $\operatorname{id} + C_0^k(\mathbb{R}^d, \mathbb{R}^d)$ and is therefore a Banach manifold. Since the composition in $C_0^k(\mathbb{R}^d, \mathbb{R}^d)$ is continuous, it is also a topological group. Moreover, for any $\psi \in C_0^k(\mathbb{R}^d, \mathbb{R}^d)$, the map

$$\begin{array}{ccc} \operatorname{Diff}_{C_0^k}(\mathbb{R}^d) & \longrightarrow & \operatorname{Diff}_{C_0^k}(\mathbb{R}^d) \\ \varphi & \longmapsto & \varphi \circ \psi \end{array}$$

is the restriction of a continuous affine map, and is therefore smooth. However, if now $\varphi \in \operatorname{Diff}_{C_0^k}(\mathbb{R}^d)$ is fixed, a formal computation shows that the derivative of $L_\varphi : \psi \mapsto \varphi \circ \psi$ is given by

$$T_\psi L_\varphi h(x) = d_{\psi(x)}\varphi(h(x)), \quad x \in \mathbb{R}^d$$

and thus involves a loss of regularity. This computation shows that the left multiplications $L_\psi$ are only continuous, meaning that the group $\operatorname{Diff}_{C_0^k}(\mathbb{R}^d)$ is not a Lie group, and is only a *half-Lie group*. Let now $l \geq 0$. If we restrict to diffeormorphisms that are $C^{k+l}$, we can however gain some regularity for the left multiplication :

**Proposition 4.1** (Regularity of the composition of diffeomorphisms). *Let $\varphi \in \operatorname{Diff}_{C_0^{k+l}}(\mathbb{R}^d)$, then left multiplication*

$$L_\varphi : \left\{ \begin{array}{ccc} \operatorname{Diff}_{C_0^k}(\mathbb{R}^d) & \longrightarrow & \operatorname{Diff}_{C_0^k}(\mathbb{R}^d) \\ \psi & \longmapsto & \varphi \circ \psi \end{array} \right.$$

*is $C^l$.*

*Proof.* We follow and adapt proofs from [16, lemma 9.9], or [70, 9]. We write $\psi = \operatorname{id} + v$ with $v \in C_0^{k+l}(\mathbb{R}^d, \mathbb{R}^d)$ so that we can consider the map

$$F_v : \left\{ \begin{array}{ccc} \operatorname{Diff}_{C_0^k}(\mathbb{R}^d) & \longrightarrow & C_0^k(\mathbb{R}^d, \mathbb{R}^d) \\ \psi & \longmapsto & v \circ \psi \end{array} \right.$$

We start with the case $l = 0$. We recall the Faa di Bruno formula (cf. [28] for one dimension, or [70, 2.5] for a multivariate version), for $\alpha \in \mathbb{N}^d_{>0}$ with $|\alpha| \leq k$

$$\partial^\alpha (v \circ \psi)(x) = \sum_{i=1}^{|\alpha|} d^i v(\psi(x)) \cdot B_{\alpha,i}(\{\partial^\beta \psi(x), |\beta| \leq |\alpha|\})$$

where $B_{\alpha,i}$ are the multivariate (symmetric) Bell polynomials. In particular, the map

$$\begin{array}{ccc} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) & \longrightarrow & C_0(\mathbb{R}^d, \mathbb{R}^d) \\ \psi & \longmapsto & \partial^\alpha (v \circ \psi) \end{array}$$

is continuous, for any $\alpha$, and thus $F$ is also continuous. Next we also prove the case $l = 1$. Let any $\delta\psi \in C_0^k(\mathbb{R}^d, \mathbb{R}^d)$, and consider the curve $c_{\delta\psi} : s \mapsto F_v(\psi + s\delta\psi) = v \circ (\psi + s\delta\psi)$. We compute the Bochner integral $\int_0^t dv_{\psi+s\delta\psi} \delta\psi ds$ in $C_0^k(\mathbb{R}^d, \mathbb{R}^d)$, and for all $x \in \mathbb{R}^d$

$$\begin{aligned} \left( \int_0^t dv_{\psi+s\delta\psi} \delta\psi ds \right)(x) &= \int_0^t dv_{\psi(x)+s\delta\psi(x)} \delta\psi(x) ds \\ &= \int_0^t \partial_s c_{\delta\psi}(s, x) ds \\ &= \left( F_v(\psi + t\delta\psi) - F_v(\psi) \right)(x) \end{aligned}$$

In particular this proves that $F_v$ is Gateaux-differentiable, and that

$$dF_v(\psi; \delta\psi) = (dv \circ \psi) \cdot \delta\psi$$

Note that the map $dF_v(\psi; \cdot)$ is linear and continuous, and thus to prove $F_v$ is actually Frechet differentiable, we use lemma 3.5 and prove that the map

$$\begin{array}{ccc} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) & \longrightarrow & L(C_0^k(\mathbb{R}^d, \mathbb{R}^d), C_0^k(\mathbb{R}^d, \mathbb{R}^d)) \\ \psi & \longmapsto & dF_v(\psi; \cdot) = dv \circ \psi \end{array}$$

is also continuous. First, we see that the map

$$\begin{array}{ccc} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) & \longrightarrow & C_0^k(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d)) \\ \psi & \longmapsto & dv \circ \psi \end{array}$$

is continuous by adapting the proof of the case $l = 0$ using Faa di Bruno formula (we recognize here the map $F_{dv}$). Moreover the injection

$$\begin{array}{ccc} C_0^k(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d)) & \longrightarrow & L(C_0^k(\mathbb{R}^d, \mathbb{R}^d), C_0^k(\mathbb{R}^d, \mathbb{R}^d)) \\ A & \longmapsto & [M_A : h \mapsto A \cdot h] \end{array}$$

is also continuous. Indeed, for any $h \in C_0^k(\mathbb{R}^d, \mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$, we get

$$|\partial^\alpha (A \cdot h)|_\infty \leq \sum_{\beta \leq \alpha} C_\beta^\alpha \|\partial^\beta A\|_\infty |\partial^{\alpha-\beta} h|_\infty$$

where $C_\beta^\alpha$ are the usual combinatorial constants. In particular this proves that

$$\|M_A\|_{op} \leq C_{k,d} |A|_{C_0^k}$$

with $C_{k,d}$ a constant depending only on $k$ and the dimension $d$. This finishes the proof for the case $l = 1$. Now for general $l \geq 2$, we proceed by induction, using the fact that the differential of $F_v$ is also a composition on the left :

$$dF_v = F_{dv}$$

with $dv \in C_0^{k+l-1}(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d))$. □

In particular, this means the multiplication

$$\begin{array}{ccc} \mathrm{Diff}_{C_0^{k+l}}(\mathbb{R}^d) \times \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) & \longrightarrow & \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) \\ \varphi, \psi & \longmapsto & \varphi \circ \psi \end{array}$$

is $C^l$, and by implicit theorem, the inverse mapping

$$\mathrm{inv} : \left\{ \begin{array}{ccc} \mathrm{Diff}_{C_0^{k+l}}(\mathbb{R}^d) & \longrightarrow & \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) \\ \varphi & \longmapsto & \varphi^{-1} \end{array} \right.$$

also becomes $C^l$.

The group $\mathrm{Diff}_{C^k}(\mathbb{R}^d)$ and the regularity properties of the composition will serve as a model example for the rest of this chapter, and most tools that we will introduce are based on this example. Note that for $M$ a finite dimensional compact manifold, we can also consider the group of $C^k$-diffeomorphisms $\mathrm{Diff}_{C^k}(M)$ which is also a half-Lie group [16]. We also mention the group of Sobolev diffeomorphisms $\mathrm{Diff}_{H^s}(M)$, with $s \geq d/2+1$ (where $M$ is a manifold with bounded geometry, and $d$ is the dimension of $M$). This space is also a half-Lie group [16, 53] and has a Hilbert structure, and was thus highly studied in the context of infinite-dimensional geometry and fluid dynamics [32]. In Arguillère's Ph.D. thesis [7, 8], LDDMM is performed on the group $\mathrm{Diff}_{H^s}(M)$, with $s \geq d/2+1$, of Sobolev diffeomorphisms on a finite-dimensional manifold $M$ of bounded geometry and dimension $d$.

## 4.2 Generalities

### 4.2.1 First definitions and differentiability conditions

The category of Banach half-Lie groups gives a natural framework to deal with differentiable groups with a Banach manifold structure. Riemannian geometries on such spaces were recently studied by Bauer, Harms, and Michor in [16]:

**Definition 4.2** (Half-Lie group)**.** A Banach (right) *half-Lie group* is a topological group with a smooth Banach manifold structure, such that the right multiplication $R_{g'} : g \mapsto gg'$ is smooth.

The main issue with dealing with half-Lie groups is that we lose the smoothness of the left multiplication and the inverse, and we therefore cannot define the classical objects from Lie group theory such as the exponential map and the adjoint representation. In order to deal with this lack of differentiability on the left, we introduce the concept of a graded family of a half-Lie groups that satify regularity conditions for the composition, following the example of the group of $C^k$ diffeomorphisms (cf. proposition 4.1). Let $\mathcal{G} = \{G^k, k \geq 0\}$ be a family of Banach half-Lie groups, and we denote simply $G = G^0$. We denote by $\mathrm{inv} : h \mapsto h^{-1}$ the inverse mapping, and for $g \in G^k$, $L_g : h \in G^k \mapsto gh \in G^k$ and $R_g : h \in G^k \mapsto hg \in G^k$ the left and right multiplications on $G^k$.

**Definition 4.3** (Admissible graded group structure)**.** We say that $\mathcal{G}$ is an *admissible graded group structure* if the following conditions are satisfied :

**(G.1)** $G^{k+1}$ is a subgroup of $G^k$ with smooth inclusion.

**(G.2)** For $l \geq 0$, the inverse mapping on the restriction $\mathrm{inv} : G^{k+l} \to G^k$ is $C^l$.

**(G.3)** For $l \geq 0$, the induced multiplication

$$\begin{array}{ccc} G^{k+l} \times G^k & \longrightarrow & G^k \\ (g', g) & \longmapsto & g'g \end{array}$$

is $C^l$ and $C^\infty$ in the first variable $g'$ for $g$ fixed.

**(G.4)** For $l \geq 0$, the induced right infinitesimal multiplication

$$\begin{array}{ccl} G^k \times T_e G^{k+l} & \longrightarrow & TG^k \\ (g, u) & \longmapsto & u \cdot g = \partial_{g'}(g'g)_{|g'=e}(u) = T_e R_g(u) \in T_g G^k \end{array}$$

is a $C^l$ mapping, and $C^\infty$ with regards to the first variable.

**(G.5)** The induced left infinitesimal action

$$\begin{array}{ccc} G^{k+1} \times T_e G^k & \longrightarrow & TG^k \\ (g, u) & \longmapsto & T_e L_g(u) \end{array}$$

is a $C^1$ vector bundle morphism.

**Remark 4.4.** *This definition is similar to the concept of ILB-Lie groups of Omori [79], except the definition of Omori imposes that $G^{k+1}$ is a dense subgroup of $G^k$ [79, Thm 3.7]. Moreover, in the context of Large deformations, we are not interested here in the properties of the limit $\bigcap_k G^k$, and we will define metrics directly on $G^k$.*

**Remark 4.5.** *Note that the property* ***(G.2)*** *is actually a consequence of* ***(G.3)*** *as we will see in the proof of proposition 4.10*

**Example 4.6** (The family $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$). *Note that, following previous section, the familly of $C^k$ diffeomorphisms $\{\mathrm{Diff}_{C_0^k}(\mathbb{R}^d),\ k > 0\}$ is in particular an admissible graded group structure.*

We have the following immediate consequences of properties **(G.3)** and **(G.5)** :

**Proposition 4.7.** *For any $k \geq 1$, $G^k$ is a topological group. Morever, we have*

1. *$R_g : G^k \to G^k$ is $C^\infty$ for any $g \in G^k$*
2. *$L_g : G^k \to G^k$ is $C^l$ for any $g \in G^{k+l}$*
3. *The left multiplication $g \in G^{k+1} \mapsto T_e L_g \in L(T_e G^k, TG^k)$ is $C^1$*
4. *The right multiplication $g \in G^k \mapsto T_e R_g \in L(T_e G^{k+1}, TG^k)$ is locally-Lipschitz, in the sense it is locally-Lipschitz in any chart.*

*Proof.* The two first points are immediate with **(G.3)**. The last point follows from **(G.4)** and by proposition 3.21, using vector bundles $G^k \times T_e G^{k+1}$ which is trivial over $G^k$ and the tangent bundle $TG^k$ of $G^k$. □

#### 4.2.1.1 Construction of admissible graded group structures with right-invariant local additions

Following constructions of [16], we can easily obtain admissible graded group structures from a half-Lie group $G$ adding an extra asumption, namely a right-invariant local addition. We sum-up these constructions here :

**Definition 4.8** (Right-invariant local addition)**.** Let $G$ be a Banach half-Lie group, and denote $\pi_G : TG \to G$ its tangent bundle. A *right-invariant local addition* on $G$ is a smooth map $\tau : V \subset TG \to G$, where $V$ is an open subset of $TG$, such that

- for every $g \in G$, we have $0_g \in V$ and $\tau(0_g) = g$, where $0_g$ denotes the zero section of $TG$
- the mapping $(\pi_G, \tau) : V \to G \times G$ is a diffeomorphism onto its range.
- for $g \in G$, $TR_g(V) = V$ and $\tau \circ TR_g = R_g \circ \tau$

**Example 4.9.** *The mapping $(x, \tau) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto x + \tau$ is a local addition on $\mathbb{R}^d$ (it is the exponential map of the canonical Riemannian metric on $\mathbb{R}^d$) and it induces the following right-invariant map on $\operatorname{Diff}_{C_0^k}(\mathbb{R}^d)$*

$$\begin{array}{ccc} T\operatorname{Diff}_{C_0^k}(\mathbb{R}^d) & \to & \mathrm{id} + C_0^k(\mathbb{R}^d, \mathbb{R}^d) \\ (\phi, u \circ \phi) & \mapsto & \phi + u \circ \phi \end{array}$$

*Since $\operatorname{Diff}_{C_0^k}(\mathbb{R}^d)$ is open in $\mathrm{id} + C_0^k(\mathbb{R}^d, \mathbb{R}^d)$, for $u \in C_0^k(\mathbb{R}^d, \mathbb{R}^d)$ close enough to $0$, the map $\phi + u \circ \phi$ is in $\operatorname{Diff}_{C_0^k}(\mathbb{R}^d)$. Therefore, there exists some subset $V \subset T\operatorname{Diff}_{C_0^k}(\mathbb{R}^d)$ such that the restriction of the previous map is a right-invariant local addition. More generally, if $M$ is compact finite dimensional manifold, equipped with a local addition (for example the exponential map of a Riemannian metric), then the group of its $C^k$ diffeomorphisms $\operatorname{Diff}_{C^k}(M)$ can be equipped with a right-invariant local addition (and similarly for $\operatorname{Diff}_{H^s}(M)$ for $s$ high enough).*

This will allow to gain some regularity for the left translations, as shown in [16]. For $k \in \mathbb{N}$, define $G^k$ as the set of $C^k$*-elements* $g \in G$ such that left translations by $g$ is $C^k$ :

$$G^k := \{g \in G,\ L_g : G \to G \text{ and } L_{g^{-1}} : G \to G \text{ are } C^k\}$$

Equivalently, denoting $\operatorname{Diff}_{C^k}(G)^G$ the set of $C^k$ right-invariant diffeomorphisms on $G$, the set $G^k$ can be defined through the identification:

$$\begin{array}{ccc} G^k & \longrightarrow & \operatorname{Diff}_{C^k}(G)^G \\ g & \longmapsto & L_g \end{array} \tag{4.1}$$

The following result [16] shows the regularity gained with such a construction :

**Proposition 4.10** (Differentiable elements)**.** *Let $G$ be a Banach half-Lie group with a right-invariant local addition, and define the family of subgroups of differentiable elements $\{G^k, k \in \mathbb{N}^*\}$ as before. Then, the groups $G^k$ are Banach half-Lie groups, and the family $\{G^k, k \in \mathbb{N}^*\}$ is an admissible group struture in the sense of 4.3:*

*Proof.* The proof is mostly included in the work of Bauer, Harms, and Michor [16, Theorem 3.4] and uses the identification 4.1. Some points are not explicitly written even though contained in the paper, so we are going to prove them for sake of completeness. We recall some of the construction of [16, §.10] giving the differentiable structure of the group $G^k \simeq \mathrm{Diff}_{C^k}(G)^G$. Let $\tau : V \subset TG \to G$ a right-invariant local addition on $G$. We define the set of $C^k(G,G)^G$ of right-invariant morphisms, that is to say $C^k$ morphisms $f : G \to G$ such that for all $g \in G$, $f \circ R_g = R_g \circ f$. We also consider the space $\Gamma_{C^k}(TG)^G$ of right invariant vector fields of $G$ (that is to say $C^k$ vector fields $X : G \mapsto TG$ such that $X \circ R_g = TR_g \circ X$). By [16, 10.7], the space $\Gamma_{C^k}(TG)^G$ is a Banach manifold. Moreover, for $f \in C^k(G,G)^G$, we define the open subset $\mathcal{V} = \{X \in \Gamma_{C^k}(TG)^G \mid X(e) \in V\}$ of $\Gamma_{C^k}(TG)^G$, and the map

$$v_f : \left\{ \begin{array}{lll} \mathcal{V} & \longrightarrow & C^k(G,G)^G \\ X & \longmapsto & \tau \circ X \circ f \end{array} \right.$$

is invertible and defines a chart $(v_f(U), v_f^{-1})$ around $f$ of $C^k(G,G)^G$ [16, 10.8]. The collection of all $\big(v_f(\mathcal{V}), v_f^{-1}\big)$ defines a Banach manifold structure on the space $C^k(G,G)^G$ modeled on the space $\Gamma_{C^k}(TG)^G$ and moreover $\mathrm{Diff}_{C^k}(G)^G$ is an open subset of $\Gamma_{C^k}(TG)^G$.

(G.1) : The fact that $G^{k+1}$ is a subgroup of $G^k$ is immediate. Moreover the inclusion is smooth as it is directly smooth in the charts.

(G.3-4): Conditions (G.3) and (G.4) follows from the fact that $G^{k+l} \subset (G^k)^l$ and using proposition 3.6.

(G.2) Therefore, for $g' \in G^{k+l}, g \in G^k$, the derivative :

$$TL_{g'} : \begin{array}{ccc} T_g G^k & \longrightarrow & T_{g'g} G^k \\ X & \longmapsto & g'X \end{array}$$

is a Banach isomorphism. Therefore, by implicit theorem the inverse map inv is $C^l$ from $G^{k+l}$ to $G^k$.

(G.5) In charts, we see that the inclusion $G^{k+1} \subset (G^k)^1$ is smooth. Moreover, by [16, lemma 10.10], the map

$$\begin{array}{ccc} (G^k)^1 & \longrightarrow & J^1(G^k, G^k) \\ g & \longmapsto & j^1_e L_g = (e, g, T_e L_g) \end{array}$$

is also smooth. In particular, for any $u \in T_e G^k$, the evaluation $g \mapsto T_e L_g u$ is smooth. $\square$

In this case, we can define the group of smooth elements $G^\infty$ by

$$G^\infty = \bigcap_{k \geq 0} G^k.$$

The group $G^\infty$ is then a Frechet Lie group [16, Lemma 3.7]. Its Lie algebra $T_e G^\infty$ is the space of vectors $X \in T_e G$ such that the induced right-invariant vector field $\tilde{X} : g \mapsto T_e R_g(X)$ is smooth.

### 4.2.2 Adjoint representation and Lie bracket

We introduce next equivalents of the adjoint representation and of the Lie bracket on half-Lie groups. Since the multiplication in the group is only partially smooth and continuous, we cannot directly define a Lie bracket in the tangent space $T_e G$. However, we consider again a family $\{G^k, k \geq 0\}$ of half-Lie groups (with $G = G^0$) satisfying (G.1-5) (for example if $G$ is a half-Lie group equipped with a right-invariant local addition). For $g \in G$, we denote by $\mathrm{int}_g$ the interior automorphism defined by

$$\mathrm{int}_g : G \to G, h \mapsto ghg^{-1}.$$

Moreover, if $g \in G^1$, the interior automorphism $\mathrm{int}_g = R_{g^{-1}} \circ L_g$ is thus $C^1$. In particular, we define the adjoint representation.

**Definition 4.11** (Adjoint representation)**.** Let $g \in G^1$. We define the *adjoint representation* $\mathrm{Ad}_g \in L(T_eG, T_eG)$ as the derivative of the interior morphism

$$\mathrm{Ad}_g = T_e \,\mathrm{int}_g$$

Following [16], we can also define an analogous of the Lie bracket. By (G.4), any element $u \in T_eG^1$ extends uniquely to a right-invariant $C^1$ vector field

$$\tilde{u}(g) = T_eR_g(u),\ \ g \in G$$

We denote by $\Gamma_{C^1}(TG)^G$ the linear space of right-invariant $C^1$ vector fields on $G$, i.e. the space of $C^1$ vector fields $\xi$ satisfying

$$\xi \circ R_g = TR_g\xi,\ \text{ for any } g \in G$$

We have in particular the inclusion

$$\begin{array}{ccc} T_eG^1 & \hookrightarrow & \Gamma_{C^1}(TG)^G \\ u & \mapsto & \tilde{u} \end{array}$$

In the particular case where $G$ is equipped with a right-invariant local addition, and $G^1$ is the half-Lie subgroup of $C^1$ elements of $G$, then this inclusion is actually an equality $T_eG^1 \simeq \Gamma_{C^1}(TG)^G$. Similarly, for $k \geq 1$, the tangent space $T_eG^k$ is naturally included in the space of right-invariant $C^k$ vector fields $\Gamma_{C^k}(TG)^G$ (and if $G^k$ is the set of $C^k$-elements of $G$, then it is a bijection $G^k \simeq \Gamma_{C^k}(TG)^G$.

**Definition 4.12** (Lie bracket definition)**.** We define a Lie bracket $[\cdot,\cdot] : T_eG^1 \times T_eG^1 \to T_eG$ by

$$[u, v] = T_e\tilde{u}(v) - T_e\tilde{v}(u)$$

where $u, v \in T_eG^1$, and the equality has to be understood in charts around $e \in G$.

**Remark 4.13.** *Here $u$ and $v$ induces $C^1$ right-invariant vector fields in $TG$ and this definition actually coincides with the usual definition of Lie brackets of vector fields in Banach manifolds as described in [61, chapter V].*

*Proof.* We prove that this definition does not depend of the charts. Let $u, v \in T_eG^1$, and $(U, \phi)$ a chart of $e$ in $G$, such that $\phi(e) = 0 \in \mathbb{B}$. This also induces a chart $(TU, d\phi)$ of $TG$. We denote by $\hat{u} = d\phi \circ \tilde{u} \circ \phi^{-1} : U \to \mathbb{B}$ and $\hat{v} = d\phi \circ \tilde{v} \circ \phi^{-1} : U \to \mathbb{B}$ the induced maps in the chart $U$. The Lie bracket in the local chart is thus defined by the formula

$$[\hat{u}, \hat{v}](x) = d_x\hat{u}(\hat{v}(x)) - d_x\hat{v}(\hat{u}(x))$$

Let $\psi : \mathbb{B} \to \mathbb{B}$ be a smooth diffeomorphism of $\mathbb{B}$, such that $\psi(0) = 0$. We also denote by $\psi_* \hat{u} = d_{\psi^{-1}(.)}\psi \circ \hat{u} \circ \psi^{-1}$, (and similarly $\psi_* \hat{v} = d_{\psi^{-1}(.)}\psi \circ \hat{v} \circ \psi^{-1}$), the pushforward of $\hat{u}$ by $\psi$. Then we get

$$\begin{aligned}
[\psi_* \hat{u}, \psi_* \hat{v}](x) &= d_x \psi_* \hat{u}(\psi_* \hat{v}(x)) - d_x \psi_* \hat{v}(\psi_* \hat{u}(x)) \\
&= d^2_{\psi^{-1}(x)}\psi(\psi_* \hat{u}(x)), \psi_* \hat{v}(x)) + \psi_* \left(d\hat{u}(\hat{v})\right)(x) \\
&- d^2_{\psi^{-1}(x)}\psi(\psi_* \hat{v}(x, \psi_* \hat{u}(x)) - \psi_* \left(d\hat{v}(\hat{u})\right)(x) \\
&= \psi_* \left(d\hat{u}(\hat{v})\right)(x) - \psi_* \left(d\hat{v}(\hat{u})\right)(x) \\
&= \psi_*[\hat{u}, \hat{v}](x)
\end{aligned}$$

where we used the fact that $d^2\psi$ is a symmetric bilinear map of $\mathbb{B}$. This concludes the proof. □

Let $p \in [1, \infty]$. We recall that the set of derivations $\mathrm{Der}(G)$ is the algebra of linear operators $\partial$ from $C^p(G, \mathbb{R})$ to $C^{p-1}(G, \mathbb{R})$ satisfying the Leibniz law, i.e.

$$\partial(fg) = \partial(f)g + f\partial(g) \tag{4.2}$$

Any vector field $\xi \in \Gamma(TG)$ defines a unique derivation $\mathcal{L}_\xi \in \mathrm{Der}(G)$ by

$$\mathcal{L}_\xi f = Tf(\xi)$$

and moreover the induced morphism $\xi \in \Gamma(TG) \mapsto \mathcal{L}_\xi \in \mathrm{Der}(G)$ is injective. Since we are dealing with right-invariant vector fields, the associated derivation is uniquely determined by its value at identity. Indeed, for any $\xi \in \Gamma_{C^1}(TG)^G$, $g \in G$ and $f \in C^\infty(G, \mathbb{R})$, we have

$$\mathcal{L}_\xi f(g) = Tf(\xi(g)) = Tf \circ TR_g(\xi(e)) = \mathcal{L}_\xi(f \circ R_g)(e),$$

with $f \circ R_g \in C^\infty(G, \mathbb{R})$, so that the map

$$\begin{array}{ccc}
\Gamma_{C^1}(TG)^G & \longrightarrow & C^\infty(G, \mathbb{R})^* \\
\xi & \longmapsto & (f \mapsto \mathcal{L}_\xi f(e))
\end{array}$$

is an injection.

**Proposition 4.14** (Lie bracket and derivation)**.** *Let $u, v \in T_eG^1$, the element $[u, v]$ is the unique element of $T_eG$ such that for any smooth function $f : G \to \mathbb{R}$,*

$$\mathcal{L}_{[\tilde{u},\tilde{v}]} f(e) = \mathcal{L}_{\tilde{v}}(\mathcal{L}_{\tilde{u}} f)(e) - \mathcal{L}_{\tilde{u}}(\mathcal{L}_{\tilde{v}} f)(e) \tag{4.3}$$

*where $Tf(u)$ and $Tf(v)$ are smooth maps from $G$ to $\mathbb{R}$ defined by $g \mapsto T_g f(\tilde{u}), T_g f(\tilde{v})$.*

*Proof.* The proof is also contained in [16, 61], even though it is not really detailed in this setting. We add it here for sake of completeness. Let $u, v \in T_eG^1$, and we consider local representations $\hat{u}$ and $\hat{v}$ in a chart $(U, \phi)$ around $e \in G$, as in the proof of proposition 4.12. For $f \in C^\infty(G, \mathbb{R})$, we also denote by $\hat{f} = f \circ \phi$ its representation in the chart $U$. We have

$$\begin{aligned}
\mathcal{L}_{\hat{v}}(\mathcal{L}_{\hat{u}} \hat{f})(0) - \mathcal{L}_{\hat{u}}(\mathcal{L}_{\hat{v}} \hat{f})(0) &= d_0(d\hat{f}(\hat{u}))(\hat{v}) - d_0(d\hat{f}(\hat{v}))(\hat{u}) \\
&= d_0^2 \hat{f}(\hat{u}, \hat{v}) + d_0 \hat{f}\left(d_0 \hat{u}(\hat{v}(0))\right) - d_0^2 \hat{f}(\hat{v}, \hat{u}) + d_0 \hat{f}\left(d_0 \hat{v}(\hat{v}(u))\right) \\
&= d_0 \hat{f}\left(d_0 \hat{u}(\hat{v}(0))\right) - d_0 \hat{f}\left(d_0 \hat{v}(\hat{u}(0))\right) \\
&= d_0 \hat{f}([\hat{u}, \hat{v}]) \\
&= \mathcal{L}_{[\hat{u},\hat{v}]} \hat{f}(0)
\end{aligned}$$

which conclude the proof. $\square$

Since any $u \in T_eG^{k+1}$ also defines a $C^1$ right-invariant vector field on $G^k$, we can similarly define a Lie bracket $[.,.]_k : T_eG^{k+1} \times T_eG^{k+1} \to T_eG^k$. This bracket product satisfies many properties of usual Lie brackets:

**Proposition 4.15** (Lie bracket)**.** *The Lie bracket $[\cdot,\cdot] : T_eG^1 \times T_eG^1 \to T_eG$ satisfies*

1. $[T_eG^{k+1}, T_eG^{k+1}] \subset T_eG^k$ *and coincides with the bracket product $[\cdot,\cdot]_k$ on $G^k$*
2. $[\cdot,\cdot]$ *is a skew-symmetric continuous bilinear mapping and satisfies a Jacobi identity:*
$$\forall u, v, w \in T_eG^2,\ \ [u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0. \tag{4.4}$$
3. *The adjoint representation* $\mathrm{Ad}$ *induces a $C^1$ mapping* $\mathrm{Ad} : G^2 \times T_eG^2 \to T_eG$ *and*
$$\forall u, v \in T_eG^2,\ \ T_e\left(\mathrm{Ad}(v)\right)(u) = [u,v]. \tag{4.5}$$
4. *The inverse of the adjoint representation* $\mathrm{Ad}^{-1}$ *also induces a $C^1$ map* $\mathrm{Ad}^{-1} : G^1 \times T_eG^1 \to T_eG$, $(g,v) \mapsto \mathrm{Ad}_{g^{-1}} v$ *and*
$$\forall u, v \in T_eG^1,\ \ T_e\left(\mathrm{Ad}^{-1}(v)\right)(u) = -[u,v]. \tag{4.6}$$

*Proof.* Let $u, v \in T_eG^{k+1}$. Then the Lie bracket $[u,v]_k \in T_eG^k$ is in particular in $T_eG$ and thus defines a derivation $\mathrm{Der}(G)$ : for $f \in C^\infty(G,\mathbb{R})$, we have

$$\mathcal{L}_{[u,v]_k} f(e) = \mathcal{L}_{[u,v]} f(e)$$

so that $[u,v]_k = [u,v]$, and we get the first point. The second point is also straightforward and follows from the definition of the Lie bracket. Let us prove the third point. By (G.2), the inverse mapping $\mathrm{inv} : G^2 \to G$ is $C^2$, and since the multiplication $(g,g') \in G^2 \times G \to gg' \in G$ is also $C^2$, the map

$$\begin{array}{ccc} G^2 \times G^2 & \longrightarrow & G \\ g, g' & \longmapsto & gg'g^{-1} \end{array}$$

is $C^2$. Thus the partial derivative

$$\begin{array}{ccc} G^2 \times T_eG^2 & \longrightarrow & G \\ g, v & \longmapsto & \mathrm{Ad}_g(v) \end{array}$$

is $C^1$. We thus define $w = T_e(\mathrm{Ad})(v)(u)$. For any $f \in C^\infty(G,\mathbb{R})$, we have

$$\begin{aligned} \partial_g \partial_{g'} f(gg'g^{-1})|_{g'=e,g=e}(v)(u) &= \partial_g \left(T_e f(\mathrm{Ad}_g(v))\right)|_{g=e}(u) \\ &= T_e f(w) \\ &= \mathcal{L}_w f(e) \end{aligned}$$

On the other hand, by Schwarz theorem we also have

$$\begin{aligned} \partial_g \partial_{g'} f(gg'g^{-1})|_{g'=e,g=e}(v)(u) &= \partial_{g'} \partial_g f(gg'g^{-1})|_{g=e,g'=e}(u)(v) \\ &= \partial_{g'} \left(Tf \circ T_e R_{g'}(u)\right)|_{g'=e}(v) + \partial_g \left(Tf \circ T_e R_{g^{-1}}(v)\right)|_{g=e}(u) \\ &= \mathcal{L}_{\tilde{v}}(\mathcal{L}_{\tilde{u}} f)(e) - \mathcal{L}_{\tilde{u}}(\mathcal{L}_{\tilde{v}} f)(e) \\ &= \mathcal{L}_{[\tilde{u},\tilde{v}]} f(e) \end{aligned}$$

so that the third point also follows.

We prove now the last point, with $g \in G^1$ and $v \in T_eG^1$. By definition, we get

$$\mathrm{Ad}_{g^{-1}} v = (T_eL_g)^{-1} T_eR_g v.$$

By (G.4), the map $G^1 \times T_eG^1 \to TG, g, v \mapsto T_eR_g v$ is $C^1$. Moreover, by (G.5), $T_eL : G^1 \times T_eG \to TG^1$ is a $C^1$ vector bundle morphism. This means in particular that

$$\begin{array}{ccc} G^1 & \longrightarrow & L(T_eG, TG) \\ g & \longmapsto & T_eL_g \end{array}$$

is $C^1$ (cf. definition 3.19). In a local chart $U$ of the bundle $TG \to G^1$ around $g \in G^1$, $g \mapsto T_eL_g$ is the map $g \mapsto (g, \widehat{T_eL_g}) \in U \times GL(T_eG)$. As $GL(T_eG)$ is a Banach Lie group, the inverse mapping $\mathrm{inv} : GL(T_eG) \to GL(T_eG)$ is smooth. Thus, the map $g \in G^1 \mapsto (T_eL_g)^{-1} \in L(TG, T_eG)$ is also $C^1$. Therefore, the map

$$\begin{array}{ccc} G^1 \times T_eG^1 & \longrightarrow & T_eG \\ (g, v) & \longmapsto & \mathrm{Ad}_{g^{-1}} v \end{array}$$

is $C^1$. The last equality follows from computations similar to those in the proof of the third point. □

In particular, following classical notations for the Lie bracket in finite dimensions, we also define $\mathrm{ad} : T_eG^1 \to L(T_eG, T_eG)$ by

$$\mathrm{ad}_u v = [u, v]$$

Moreover if $G$ carries a right-invariant local addition, and $G^\infty$ is the Lie group of smooth elements, this bracket becomes a true Lie bracket on $T_eG^\infty$ and the space becomes a Lie algebra [16].

## 4.3 Regularity and evolution map

In this part, we deal with regularity properties of half-Lie groups, in the sense that under some regularity conditions, there will exist a unique global flow associated with a right-invariant vector field :

**Definition 4.16** (Regularity of a group). Let $G$ be a Banach half-Lie group. We say that the group $G$ is *$L^p$-regular* if for any $u \in L^p(\mathbb{R}, T_eG)$, there exists a unique curve $g \in AC_{L^p}(\mathbb{R}, G)$ that is solution of the differential equation

$$\dot{g}(t) = T_eR_{g(t)} u(t), \quad g(0) = e.$$

Let $\{G^k,\ k \geq 0\}$ be a family of Banach half-Lie groups satisfying (G.1-5) 4.3, and denote $G = G^0$. Let $I \subset \mathbb{R}$ be a compact interval, and suppose $G^k$ is modeled on the Banach space $\mathbb{B}^k$. The regularity of Banach half-Lie groups was proved in [16, Theorem 4.2] when restricting to smooth curves in the tangent space of identity. In our setting, we consider less regular curves with $AC_{L^2}$-regularity, which generates absolutely continuous curves in $G^k$. Let $AC_{L^p}(I, G^k)$ (resp. $AC_{L^p}(I, G)$) be the set of absolutely continuous curves in $G^k$ (resp. in $G$). We recall the Banach differential structure of $AC_{L^p}(I, G)$.

**Proposition 4.17** (Banach manifold structure of $AC_{L^p}(I,G)$)**.** *Assume $I=[a,b]$ with $a<b\in\mathbb{R}$.*

1. *The space $AC_{L^p}(I,G^k)$ is a Banach manifold.*
2. *For $t\in I$, the evaluation*
$$\mathrm{ev}_t : \begin{cases} AC_{L^p}(I,G) & \longrightarrow & G \\ \eta & \longmapsto & \eta(t) \end{cases}$$
*is smooth.*
3. *The tangent bundle $TAC_{L^p}(I,G)$ is identified with the vector bundle $AC_{L^p}(I,TG)\to AC_{L^p}(I,G)$. For $g\in AC_{L^p}(I,G)$, we therefore have*
$$T_gAC_{L^p}(I,G)=AC_{L^p}(I\leftarrow g^*TG)$$
*where $AC_{L^p}(I\leftarrow g^*TG)=\{\gamma\in AC_{L^p}(I,TG),\ \gamma(t)\in T_{g(t)}G,\ \forall t\in I\}$.*

*Proof.* This follows directly from Theorem 3.37. □

The rest of this section is devoted to the study of absolutely continuous curves in $G^k$. We start by showing the existence of an absolutly continuous lift in $G^k$ for any integrable curve in the tangent space $T_eG^{k+1}$.

**Proposition 4.18** (Regularity of Banach half-Lie groups)**.** *Let $k\geq 0$, $I\subset\mathbb{R}$ be an open interval and $t_0\in I$. For every $u\in L^p(I,T_eG^{k+1})$ the ordinary differentiable equation*
$$\begin{cases} \dot g(t)=u(t)\cdot g(t)=T_eR_{g(t)}(u(t)) \\ g_{t_0}=e \end{cases} \tag{4.7}$$
*admits a unique global (i.e. defined on $I$) solution $g\in AC_{L^p}(I,G^k)$.*
*Moreover, if $u\in L^p(I,T_eG^{k+n})$, for $n\geq 1$, then for all $t\in I$, the left translation $L_{g(t)}:G^k\to G^k$ is $C^n$.*

*Proof.* The proof is contained in [82], and in [16] for the special case where $u\in C^\infty(I,T_eG^k)$. The first step is to prove local existence of solutions, using classic Picard-Lindelof theorem. Let $(U,\varphi)$ be a local chart around $e$ in $G^k$ with $\varphi:U\to\mathbb{B}^k$ where $\mathbb{B}^k$ is the modelling Banach space for $G^k$. We denote $V=\varphi(U)\subset\mathbb{B}^k$ and $d_g\varphi\doteq\mathrm{pr}_2\circ T_g\varphi$ for any $g\in U$ where $\mathrm{pr}_2:V\times\mathbb{B}^k\to\mathbb{B}^k$ the canonical projection on the second argument. We consider $f:V\times\mathbb{B}^{k+1}\to\mathbb{B}^k$ such that
$$f(y,v)=d_g\varphi(u\cdot g)\text{ with }(u,g)=((d_e\varphi)^{-1}(v),\varphi^{-1}(y))\,.$$
From (G.4), $f$ is $C^1$. Since $(y,v)\mapsto\partial_yf(y,v)$ is continuous on $V\times\mathbb{B}^{k+1}$ and linear in $v$, we can assume (up to the restriction of $V$ to a smaller open set) that there exists $K>0$ such that $\|\partial_yf(y,v)\|_{L(\mathbb{B}^k,\mathbb{B}^k)}\leq K|v|_{\mathbb{B}^{k+1}}$ for any $(y,v)\in V\times\mathbb{B}^{k+1}$. Hence the mapping $F:I\times\varphi(U)\longrightarrow\mathbb{B}^k$ defined by $F(t,y)=f(y,v_t)$ where $v_t=d_e\varphi(u_t)$ is such that
$$|F(t,y)-F(t,y')|_{\mathbb{B}^k}\leq K|v_t|_{\mathbb{B}^{k+1}}|y-y'|_{\mathbb{B}^k}$$
so that $F(t)\doteq F(t,.)\in\mathrm{Lip}(V,\mathbb{B}^k)$ and, since $t\mapsto v_t$ is $L^p(I,\mathbb{B}^{k+1})$ for $u\in L^p(I,T_eG^{k+1})$, $t\mapsto F(t)\in L^1_{loc}(I,\mathrm{Lip}(V,\mathbb{B}^k))$. Therefore by [98, Theorem C.6], for any $t_0\in I$, the

equation 4.7 that is locally equivalent to $\dot{y}_t = F(t, y_t)$ has one unique local solution in $G^k$ defined on $I_{t_0} \subset I$. In the case where $u \in L^p(I, T_e G^{k+n})$ for some $n \geq 1$, we even have that $t \mapsto F(t) \in L^1_{loc}(I, C^n_b(V, \mathbb{B}^k))$. Therefore, by [98, Theorem C.15 and C.18] for $t \in I_{t_0}$, the mapping $y_t : x \in V \mapsto y_t(x) \in \mathbb{B}^k$ is $C^n$, where $y_t(x)$ is solution of the equation $\dot{y}_t = F(t, y_t)$ with initial point $y_{t_0} = x$. By uniqueness of the solution, this corresponds in $G^k$ to a $C^n$ mapping $g \in U \mapsto g(t)g = L_{g(t)}(g) \in G^k$ so that $L_{g(t)}$ is locally $C^n$ on $U$. Since $R_h(U) = Uh = R^{-1}_{h^{-1}}(U)$ is an open neighborhood of $h \in G^k$ and since for $y \in R_h(U)$, $L_{g(t)}(y) = R_h \circ L_{g(t)} \circ R_{h^{-1}}(y)$ where $R_h$ and $R_{h^{-1}}$ are smooth mappings (see (G.3)), we get that $L_{g(t)}$ is $C^n$ on $G^k$ for all $t \in I_{t_0}$.

Now, following the proof from [59, 41], we prove that the solution is globally defined on $I$. It is enough to consider the case where $I = [a, b]$ is compact. We get from the above local existence of solutions, i.e. there exists $a = a_1 < a_2 < \ldots < a_n = b$ such that $I = [a_1, a_n]$ and, for any $1 \leq i < n$ there exists solution $g_i \in AC_{L^p}([a_i, a_{i+1}], G^k)$ of

$$\begin{cases} \dot{g}_i(t) = u(t) \cdot g_i(t) \\ g_i(a_i) = e\,. \end{cases}$$

We can now define a global solution $g(t) = g_i(t) g_{i-1}(a_i) \ldots g_1(a_2) = R_{g_{i-1}(a_i) \ldots g_1(a_2)}(g_i(t))$ for $a_i \leq t \leq a_{i+1}$ which is in $AC_{L^p}(I, G^k)$ as $R_{g_{i-1}(a_i) \ldots g_1(a_2)}$ is smooth and each $g_i$ is $AC_{L^p}$. $\square$

In particular, for half-Lie groups equipped with right-invariant local addition, we can prove that the subgroups of differentiable elements are regular.

**Corollary 4.19** (Regularity of the group of $C^k$-differentiable elements)**.** *Let $G$ be a Banach half-Lie group that carries a right-invariant local addition, and let $G^k$ denotes the group of $C^k$-differentiable elements. Then for $k \geq 1$, and potentially $k = \infty$, the group $G^k$ is $L^p$-regular.*

*Proof.* This follows directly from 4.18. Indeed, for any $u \in L^p(\mathbb{R}, T_e G^k)$, with $k \geq 1$, we can integrate equation (4.7) in $G$ and get a solution $g \in AC_{L^p}(\mathbb{R}, G)$. Moreover for any $t \in \mathbb{R}$, the map

$$L_{g(t)} : G \to G$$

is $C^k$, i.e. $L_{g(t)} \in \mathrm{Diff}_{C^k}(G)^G$. In charts defined in 4.10, we see that the map $t \mapsto L_{g(t)}$ is in $AC_{L^p}(I, \mathrm{Diff}_{C^k}(G)^G)$ and that

$$\frac{d}{dt} L_{g(t)} = L_{u(t)} \circ L_{g(t)}$$

where $L_{u(t)} \in \Gamma_{C^k}(TG)^G = T_{\mathrm{id}} \mathrm{Diff}_{C^k}(G)^G$ is defined by $L_{u(t)} g = T_e R_g u(t)$. Therefore the curve $t \mapsto g(t) = L_{g(t)}(e)$ is in $AC_{L^p}(\mathbb{R}, G^k)$ and satisfies (4.7). $\square$

**Remark 4.20** ($L^p$-regularity of $\mathrm{Diff}_{C^1}(M)$)**.** *Note that proposition 4.19 just proves $L^p$-regularity for the half-Lie group $G^k$ where $k \geq 1$. The case $k = 0$, i.e. the $L^p$-regularity of $G$ can be a bit trickier, and the example of the group of diffeomorphisms was particularly studied. We refer to [19] for a proof of $L^1$-regularity for the group $\mathrm{Diff}_{H^s}(M)$, where $M$ is a manifold of dimension $d$ of bounded geometry and $s > d/2 + 1$. A recent work [83] also prove the $L^p$-regularity of the half-Lie group $\mathrm{Diff}_{C^1}(M)$.*

For the end of the section, let $\{G^k,\ k \geq 0\}$ an admissible graded group structure, with $G = G^0$. Proposition 4.18 allows to define the evolution map for any time-dependent vector field $u \in L^p(I, T_eG^{k+1})$.

**Definition 4.21** (Evolution map)**.** We denote by $\mathrm{Evol}_{G^k} : L^p(I, T_eG^k) \to AC_{L^p}(I, G^k)$ the *evolution map* associating to any time-dependent vector field $u \in L^p(I, T_eG^k)$ the solution $g \in AC_{L^p}(I, G^k)$ of the ODE 4.7.

We can prove that the evolution map has some regularity, depending on the space where $u$ lives:

**Proposition 4.22** (Derivative of the evolution map)**.** *Suppose $I$ is a compact interval.*

1. *The restriction of the evolution* $\mathrm{Evol}_{G^k} : L^p(I, T_eG^{k+1+l}) \to AC_{L^p}(I, G^k)$ *with* $l \geq 0$ *is* $C^l$.
2. *For* $u, \delta u \in L^p(I, T_eG^{k+1+l})$*, its derivative* $\delta g = T_u \mathrm{Evol}_{G^k}(\delta u) \in AC_{L^p}(I \leftarrow g^*TG^k)$ *is the unique solution of the linear Cauchy problem:*

$$\delta\dot{g}(t) = \partial_g(T_eR_g u(t))|_{g=g(t)}\delta g(t) + \partial_u(T_eR_{g(t)}u)|_{u=u(t)}\delta u(t),\ \delta g(0) = 0 \tag{4.8}$$

*where* $g(t) = \mathrm{Evol}_{G^k}(u)(t)$.

*Proof.* The proof uses ideas from [8, 41]. We assume $I = [0, 1]$ and we prove the result locally. Let $u_0 \in L^p(I, T_eG^{k+l+1})$, $g_0 = \mathrm{Evol}_{G^k}(u_0)$ and consider a chart $(\mathcal{U} = AC_{L^p}(a; U), \Phi)$ around $g_0$ as introduced in 3.38 where $a = (a_i)_{1\leq i\leq n}$, $U = (U_i)_{1\leq i\leq n}$ and $\phi = (\varphi_i)_{1\leq i\leq n}$ are such that $0 = a_0 < a_1 < \ldots < a_n = 1$, and $(U_1, \varphi_1), \ldots, (U_n, \varphi_n)$ is a collection of charts on $G^k$. In the sequel we denote $I_i = [a_{i-1}, a_i]$ and $V_i = \varphi_i(U_i)$ for $1 \leq i \leq n$.

Working in local coordinates, consider for $1 \leq i \leq n$ the mapping $V_i \times T_eG^{k+l+1} \to \mathbb{B}^k$ defined by $(y^i, u) \mapsto u \cdot y^i \doteq d_g\varphi_i(u \cdot g) = d_e(\varphi_i \circ R_g)(u)$ with $g = \varphi_i^{-1}(y^i)$. From $(G.4)$ we get that this mapping is $C^{l+1}$ and therefore induced a mapping $AC_{L^p}(I_i, V_i) \times L^p(I_i, T_eG^{k+l+1}) \to L^p(I_i, \mathbb{B}^k)$ defined by $(y^i, u) \mapsto u\cdot y^i \doteq (t \mapsto u(t)\cdot y^t(t))$ which is $C^l$ [41, prop 2.3] so that we get eventually a $C^l$ mapping $(\prod_{i=1}^n AC_{L^p}(I_i, V_i)) \times L^p(I, T_eG^{k+l+1}) \to \prod_{i=1}^n L^p(I_i, \mathbb{B}^k)$ defined by $(y, u) \mapsto u \cdot y \doteq (u.y^i)_{1\leq i\leq n}$ for $y = (y^i)_{1\leq i\leq n}$.

We consider now the $C^l$ mapping

$$C : \begin{cases} \prod_{i=1}^n AC_{L^p}(I_i, V_i) \times L^p(I, T_eG^{k+1+l}) & \to & \mathbb{B}^k \times \prod_{i=1}^n L^p(I_i, \mathbb{B}^k) \times (\mathbb{B}^k)^{(n-1)} \\ (y, u) & \mapsto & (y^1(0), \dot{y} - u \cdot y, \sigma(y)) \end{cases}$$

where $\sigma(y) = (\varphi_{i+1} \circ (\varphi_i)^{-1} \circ y^i(a_i) - y^{i+1}(a_i))_{1\leq i<n}$ is the smooth mapping introduced in 3.38 checking for the continuity conditions at the boundaries of the segments $I_i$ so that if $y_{\mathrm{ev}}(u) = \Phi(\mathrm{Evol}_{G^k}(u))$ for $u \in L^p(I, T_eG^{k+1+l})$, then $y_{\mathrm{ev}}(u)$ verifies :

$$C(y_{\mathrm{ev}}(u), u) = (e_{G^k}, 0, 0)$$

However,

$$\partial_y C\,(y, u)\,\delta y = \big(\delta y^1(0), \delta\dot{y} - \partial_y(u \cdot y)\delta y, d_y\sigma(\delta y)\big)$$

and for any $(\delta q_0, \delta w, \delta\sigma) \in \mathbb{B}^k \times \prod_{i=1}^n L^p(I_i, \mathbb{B}^k) \times (\mathbb{B}^k)^{(n-1)}$, the equation

$$\partial_y C(y, u)\delta y = (\delta q_0, \delta w, \delta\sigma)$$

is a linear Cauchy problem on each segment $I_i$ with boundary conditions induced by $\delta q_0$ and $\delta\sigma$ at time $(a_i)_{0\leq i\leq n}$ that admits a unique global solution $\delta y \in \prod_{i=1}^n AC_{L^p}(I_i, V_i)$. Thus $\partial_y C(y,u) : \prod_{i=1}^n AC_{L^p}(I_i, V_i) \to \mathbb{B}^k \times \prod_{i=1}^n L^p(I_i, \mathbb{B}^k) \times (\mathbb{B}^k)^{(n-1)}$ is a Banach isomorphism and by implicit function theorem, there exists an open neighborhood $W_0 \subset L^p(I, T_eG^{k+1+l})$ of $u_0$ such that $y_{\text{ev}}$ and $\text{Evol}_{G^k} : W_0 \to AC_{L^p}(a; U)$ are $C^l$.

Moreover, $\delta y = T_u y_{\text{ev}}(\delta u)$ is solution of $\partial_y C(y,u)\delta y + \partial_u C(y,u)\delta u =$ i.e. $\delta y^1(0) = 0$, $d_y\sigma(\delta y) = 0$ and

$$\delta\dot{y} = \partial_u(u\cdot y)\delta u + \partial_y(u\cdot y)\delta y = 0$$

which is again a linear Cauchy problem on each segment $I_i$ with boundary conditions induced by $\delta y^1(0) = 0$ and $\delta\sigma = 0$ at $t = a_i$ for $0 \leq i < n$ and admitting a unique global solution. Thus, for $g = \text{Evol}_{G^k}(u)$, we get that $\delta g$ is solution of (4.8). □

## 4.4 Split extensions and examples

In this section, we review some examples of half-Lie groups, and that we will use throughout this thesis. Most of those examples are extensions of the diffeomorphism group. We simply present them without focusing on their differentiable structures, which will be addressed in the next sections.

**Diffeomorphism group** As we saw before the group $\text{Diff}_{C^k}(M)$ of diffeomorphisms of finite regularity of a compact manifold $M$ is a half-Lie group that carries a right-invariant local addition. Moreover, the subgoup of $C^l$-differentiable elements is directly $\text{Diff}_{C^k}(M)^l = \text{Diff}_{C^{k+l}}$, and in particular, for all $k \geq 2$, the group $\text{Diff}_{C^k}(M)$ is $L^p$-regular with $1 \leq p < \infty$ using proposition 4.18. The case $k = 1$ is bit trickier, we refer to [83, Theorem 6.5] for a proof of the $L^p$-regularity of $\text{Diff}_{C^1}(M)$.

**Isometries and diffeomorphisms** In classical LDDMM and shape analysis, a first step is often performed to align source and target objects through rigid motions. In the multi-scale setting [71, 82], one can include the rigid motion alignment as a coarse first layer. In [18], authors develop an equivalent formulation of multi-scale registration through an iterated semi-direct product. We adapt their idea to include the rigid motion. We introduce the group of affine isometries of $\mathbb{R}^d$

$$\text{Isom}(\mathbb{R}^d) = \mathbb{R}^d \ltimes SO(d)$$

with the product law given by $(R,\tau)(R',\tau') = (RR', \tau + R\tau')$. This allows us to associate any element $(R,\tau)$ of the Lie group $\text{Isom}(\mathbb{R}^d)$ with the rigid deformation $(R,\tau) : x \in \mathbb{R}^d \mapsto Rx + \tau$. We consider the following right action of $\text{Isom}(\mathbb{R}^d)$ on $\text{Diff}_{C_0^k}(\mathbb{R}^d)$ by conjugation,

$$S \star \varphi := S^{-1}\varphi S,$$

where $S = (\tau, R) \in \text{Isom}(\mathbb{R}^d)$. This action is well defined, continuous, and we can thus consider the group $\text{Isom}(\mathbb{R}^d) \ltimes \text{Diff}_{C_0^k}(\mathbb{R}^d)$ with the following composition law:

$$(S,\varphi)\cdot(S',\varphi') := (SS', S'^{-1}\varphi S' \circ \varphi'). \tag{4.9}$$

This group is a half-Lie group and can be equipped with a right-invariant local addition. We detail this proof in section 7.2 and give other similar groups of the form $G \ltimes \text{Diff}_{C_0^k}(\mathbb{R}^d)$ where $G$ is a finite dimensional Lie group acting by smooth automorphisms on $\text{Diff}_{C_0^k}(\mathbb{R}^d)$.

**Multiscale diffeomorphism group** In [71], authors proposed a multi-scale version of LDDMM, where the data studied exists simultaneously at different scales with interactions between them. Therefore they consider that the transport of these type of shapes is done by a family of diffeomorphisms, each diffeomorphism represent a transport at different scale. Therefore they introduce the direct product

$$\prod_{1\leq l\leq L} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d),$$

where $L \geq 0$ is the number of scales. More details on these groups will be developped in section 7.3.

**Split extensions of** Diff Let $G$ be a finite dimensional Lie groups, and consider $C^k_{e_G}(\mathbb{R}^d, G)$ the space of $C^k$ maps from $\mathbb{R}^d$ to $G$ that tend to $e_G$ at infinity, and such that the derivatives up to $k$ tend to 0. In particular, we can equip $C^k_{e_G}(\mathbb{R}^d, G)$ with the pointwise multiplication of $G$ which turns it into a Lie group : For $k, k' \geq 1$, we then define the semi-direct product

$$\mathrm{Diff}_{C_0^k}(\mathbb{R}^d) \ltimes C^{k'}_{e_G}(\mathbb{R}^d, G)$$

with product law given by

$$(\varphi, \alpha).(\varphi', \alpha') = (\varphi \circ \varphi', \alpha \circ \varphi' \star_G \alpha)$$

where $\star_G$ is the composition in $G$. The group $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d) \ltimes C^{k'}_{e_G}(\mathbb{R}^d, G)$ is then a Banach half-Lie group. It was in particular introduced in [95] in the metamorphosis framework. In particular, with $G = (\mathbb{R}^d, +)$, this allows to conjugate transport of shapes with change of intensity.

**Automorphisms of a vector bundle** Let $E \to M$ be a finite dimensional vector bundle, over a compact manifold $M$. We recall that a $C^k$ vector-bundle morphism $(\underline{f}, f)$ is a pair of $C^k$ maps $\underline{f} : M \to M$ and $f : E \to E$ such that $f$ is fiber-preserving, linear on each fiber and the following diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{\underline{f}} & M \end{array}$$

commutes. The group of $C^k$-automorphisms $\mathrm{Aut}_{C^k}(E)$ is the space of $C^k$ vector bundle morphisms that are invertible. We prove in section 9.1.1 that this space is also a Banach half-Lie group.

**Automorphisms of a principal bundle** In particular, if $M$ is a compact manifold, the tangent bundle $TM$ is a vector bundle over $M$. It is therefore natural to consider the group $\mathrm{Aut}_{C^k}(TM)$ of $C^k$ automorphisms of $TM$. Moreover, this group acts naturally by pushforward on the space of metrics $\mathrm{Met}_{C^k}(M)$, as we study in section 9.2.

## 4.5 Almost Lie-Poisson structure and Lie-Poisson reduction

In this part, we introduce an analoguous of the Lie-Poisson bracket on the Lie algebra of a half-Lie group, and study the reduction of the dynamics of right-invariant Hamiltonians on the half-Lie group. This was highly studied ([65] for example) for right-invariant Hamiltonians on finite-dimensional Lie groups leading to the standard Lie-Poisson and Euler-Poincaré reductions (see also [64]).

### 4.5.1 Poisson algebra

In infinite dimension, it gets a little bit tricky to define Poisson structures (compared to the finite dimensional case) as the spaces we are dealing with are often non-reflexive and the symplectic form and Poisson brackets are therefore only weak [77, 76]. We also refer to [43] for an examination of various approches to define Poisson structures on Banach spaces.

**Definition 4.23** (Poisson algebra)**.** A *Poisson algebra* $(\mathcal{A}, \{\cdot, \cdot\})$ is a commutative associative algebra $\mathcal{A}$ endowed with a bilinear map $\{\cdot, \cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying the following properties

1. Anti-symmetry : $\forall f, g \in \mathcal{A},\ \{f, g\} = -\{g, f\}$
2. Jacobi identity : $\forall f, g, h \in \mathcal{A},\ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
3. Leibniz rule : $\forall f, g, h \in \mathcal{A},\ \{fg, h\} = f\{g, h\} + \{f, h\}g$

The map $\{\cdot, \cdot\}$ is called a *Poisson bracket*.

We also define the notion of a weak Poisson algebra following [76], on general manifolds modeled on locally convex spaces (and not only Banach spaces as before). We refer to [87] for definitions and in-detail study of infinite dimensional geometry beyond Banach spaces. In particular, we will consider Gâteaux differentiability as in [54].

**Definition 4.24** (Weak Poisson manifold)**.** Let $M$ be a smooth manifold modeled on a locally convex space. A *weak Poisson structure* on $M$ is unital subalgebra $\mathcal{A}$ of $C^\infty(M, \mathbb{R})$ (i.e. it contains the constant functions) endowed with a bilinear map $\{\cdot, \cdot\}$ such that

1. $(\mathcal{A}, \{\cdot, \cdot\})$ is a Poisson algebra.
2. For every $m \in M$, and $v \in T_m M$ satisfying $d_m f(v) = 0$ for every $f \in \mathcal{A}$, we have $v = 0$.
3. For every $h \in \mathcal{A}$, there exists a vector field $X_h$, called the *Hamiltonian vector field* of $h$, such that
$$\forall f \in \mathcal{A},\ \mathcal{L}_{X_h} f = \{f, h\}.$$

The triple $(M, \mathcal{A}, \{\cdot, \cdot\})$ is called a *weak Poisson manifold*.

**Remark 4.25.** *Here the space $C^\infty(M, \mathbb{R})$ denotes the space of Gâteaux-smooth functions. In case $M$ is Banach manifold, this set coincides with the space of Frechet smooth functions.*

**Example 4.26** (Weak Poisson structure associated with a symplectic manifold)**.** *Symplectic manifolds provide examples of Poisson structures. In particular, let $\mathcal{Q}$ be a Banach smooth manifold, and consider the canonical weak symplectic structure on the cotangent bundle $(T^*\mathcal{Q}, \omega)$ as in section 3.1.3. We can then associate to this symplectic structure a Poisson algebra $(\mathcal{A}_{T^*\mathcal{Q}}, \{\cdot, \cdot\}_\omega)$ as in [76], where $\mathcal{A}_{T^*\mathcal{Q}}$ is the subalgebra of $C^\infty(T^*\mathcal{Q}, \mathbb{R})$ defined by*

$$\mathcal{A}_{T^*\mathcal{Q}} := \{H \in C^\infty(T^*\mathcal{Q}),\ H \textit{ admits a symplectic gradient}\}.$$

*and*

$$\{F, G\} = \omega(X_F, X_G) = dF(X_G) = -dG(X_F)$$

*where $X_F$ and $X_G$ are the symplectic gradients of $F$ and $G$. This Poisson algebra satisfies conditions 1. and 2. of definition 4.24. To get an actual weak Poisson structure on $T^*\mathcal{Q}$, one needs to impose an extra condition on the weak symplectic form to get the separity condition of definition 4.24. In particular, strong symplectic Banach manifolds define a weak Poisson structure. We refer to [76, proposition 2.18] for a proof of this result.*

**Remark 4.27.** *Generalized Poisson manifold [97, 44] Note that such structures do not necesarily imply existence of a corresponding Poisson tensor, i.e. a smooth skew-symmetric tensor $\pi \in \Gamma(\bigwedge^2 TM)$ satisfying the Jacobi identity, and such that*

$$\{f, g\} = \pi(df, dg).$$

*Another approach, introduced for example in [97, 44] defines Poisson structures using Poisson tensors defined on a subbundle $\mathbb{F}$ ot $T^*M$ that is in duality with the tangent bundle $TM$. Such triple $(M, \mathbb{F}, \pi)$ is then called a* generalized Poisson manifold*. Note that the results presented in this section could be also done in this context of generalized Poisson manifolds*

We also define the notion of Poisson map between weak Poisson manifolds, i.e. morphisms of Poisson manifolds. We recall that for any smooth map $\varphi : M \to N$ between smooth manifolds $M$ and $N$, we can define the pullback $\varphi^* : C^\infty(N, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ by

$$\varphi^* F = F \circ \varphi,\ F \in C^\infty(N, \mathbb{R})$$

**Definition 4.28** (Poisson map)**.** Let $(M, \mathcal{A}_M, \{\cdot, \cdot\}_M)$ and $(N, \mathcal{A}_N, \{\cdot, \cdot\}_N)$ weak Poisson manifolds. A smooth map $\varphi : M \to N$ is called a *Poisson map* if $\varphi^*\mathcal{A}_N \subset \mathcal{A}_M$, and $\varphi^*\{F, G\}_N = \{\varphi^* F, \varphi^* G\}_M$ for any $F, G \in \mathcal{A}_N$.

We recall next the equation of motion in Poisson bracket form. We suppose for simplicity that the manifold is modeled on a Banach space so that any Hamiltonian vector field admits a local flow.

**Proposition 4.29** (Equation of motion)**.** *Let $(M, \mathcal{A}, \{\cdot, \cdot\})$ be a weak Poisson Banach manifold, and let $\varphi_t : M \to M$ be the flow of a Hamiltonian vector field associated with a Hamiltonian $H \in \mathcal{A}$, i.e.*

$$\dot{\varphi}_t = X_H \circ \varphi_t \quad \varphi_0 = \mathrm{id}_M \tag{4.10}$$

*Then we have*

1. *For all $F \in \mathcal{A}$, we have $\frac{d}{dt} F \circ \varphi_t = \{F \circ \varphi_t, H\}$, or in short*

$$\dot{F} = \{F, H\} \tag{4.11}$$

2. *$\varphi_t$ is a Poisson map, and $H \circ \varphi_t = H$*

### 4.5.2 Lie-Poisson reduction

It is known that the dual of any finite-dimensional Lie algebra carries a natural Poisson structure called the Lie-Poisson structure. In case of a Lie algebra of a given finite dimensional Lie group, this Lie-Poisson bracket is derived using the Poisson bracket associated with the cotangent bundle of the Lie group. This process is called the Lie Poisson reduction. We present here and extend this process of Lie-Poisson reduction in the setting of half-Lie groups. Let $G$ be a half-Lie group carrying a right-invariant local addition. Let then $G^k$ be the half-Lie group of $C^k$-differential elements of $G$. The issue here is that the tangent space at identity $T_eG$ of $G$ is not a Lie algebra: it is not stable by the Lie brackets defined previously. As previously, the cotangent bundle $T^*G$ is naturally endowed with a weak symplectic structure by the canonical symplectic form $\omega$, and we can then associate to this symplectic form a Poisson algebra $(\mathcal{A}_{T^*G}, \{\cdot, \cdot\}_\omega)$, where

$$\mathcal{A}^\infty_{T^*G} \coloneqq \{H \in C^\infty(T^*G),\ \text{H admits a symplectic gradient}\}$$

is a subalgebra of $C^\infty(T^*G)$. We recall that for $F, H \in \mathcal{A}^\infty_{T^*G}$, we get

$$\{F, H\}_\omega = \omega(X_F, X_H) = dH(X_F) = -dF(X_H).$$

The idea of the Lie-Poisson reduction process is to focus on right-invariant Hamiltonian maps and use this right invariance to derive an other formulation for the Hamiltonian equations.

**Definition 4.30** (Right-invariant map)**.** A map $H : T^*G \to \mathbb{R}$ is said to be *right-invariant* if for all $g \in G$, we have

$$H \circ T^*R_g = H$$

**Remark 4.31.** *In particular any right-invariant map $H : T^*G \to \mathbb{R}$ is characterized by its restriction $h = H|_{T_e^*G}$ to the dual space $T_e^*G$, by the relation*

$$H(g, p) = h(T_eR_g^* p)$$

Now, we can also endow the dual of the tangent space $T_e^*G^\infty$ with a weak Poisson structure. We define the sub-algebra

$$\mathcal{A}_{T_e^*G^\infty} \coloneqq \{h \in C^\infty(T_e^*G^\infty),\ dh \in C^\infty(T_e^*G^\infty, T_eG^\infty)\}.$$

We recall that here the space $T_eG^\infty$ is the space of vectors $X \in T_eG$ such that the induced right-invariant vector field $\tilde{X} : g \mapsto T_eR_g(X)$ is smooth. This makes the space $T_eG^\infty$ stable by the bracket defined in section 4.2. Note that $T_eG^\infty \subset T_eG$ is a Fréchet space and we equip its dual $T_e^*G^\infty$ with the Hausdorff locally convex weak-* topology, so that $T_eG^\infty$ identifies with the bidual $T_e^{**}G^\infty$. We can thus define the Lie-Poisson bracket

$$\{h, f\}_+(m) := \big(m \mid [dh(m), df(m)]\big)$$

for any $h, f \in \mathcal{A}_{T_e^*G^\infty}$, $m \in T_e^*G^\infty$. Each $f \in \mathcal{A}_{T_e^*G^\infty}$ admits a corresponding Hamiltonian vector field $\hat{X}_f \in C^\infty(T_e^*G^\infty, T_e^*G^\infty)$ such that $\hat{X}_f \cdot h = \{h, f\}_+$ for any $h \in \mathcal{A}^\infty_{T_e^*G^\infty}$, and given by:

$$\hat{X}_f(m) = -\operatorname{ad}^*_{df(m)} m.$$

The classical result from reduction theory [65, 76, 77] gives that the momentum map

$$m : \begin{cases} T^*G^\infty & \longrightarrow & T_e^*G^\infty \\ (g, p) & \longmapsto & T_eR_g^*p \end{cases}$$

is a Poisson map between the weak Poisson structures $(T^*G^\infty, \mathcal{A}^\infty_{T^*G^\infty}, \{\cdot, \cdot\}_\omega)$ and $\big(T_e^*G^\infty, \mathcal{A}_{T_e^*G^\infty}, \{\cdot, \cdot\}_+\big)$. This means that for any right-invariant mappings $H, F \in \mathcal{A}^\infty_{T^*G}$, the following equality holds:

$$\{H, F\}_\omega = \{h, f\}_+ \circ m \tag{4.12}$$

where $h, f \in \mathcal{A}_{T_e^*G^\infty}$ are such that $h \circ m = H$ and $f \circ m = F$. In particular, this implies that, if $(g, p) \in C^\infty(I, T^*G)$ is the flow of a Hamiltonian vector field $X_H = \nabla^\omega H$ where $H = h \circ m \in \mathcal{A}_{T_e^*G^\infty}$ is right-invariant, the momentum $m_t = m(g_t, p_t) \in T_eG^\infty$ is the flow of $\hat{X}_h$, i.e. solves the reduced Lie-Poisson equation:

$$\dot{m}_t + \operatorname{ad}^*_{dh(m)} m = 0. \tag{4.13}$$

The Hamiltonian functions that we are going to introduce on $T^*G$ in the next sections are not smooth and do not belong to $\mathcal{A}^\infty_{T^*G}$, and momentum maps have to be in $T_e^*G^k$, so this construction is thus not applicable. However, these Poisson brackets can be extended on the algebras

$$\mathcal{A}^l_{T^*G} := \{H \in C^l(T^*G),\ \text{H admits a symplectic gradient}\}$$

and

$$\mathcal{A}_{T_e^*G^l} := \{h \in C^\infty(T_e^*G^l),\ dh \in C^\infty(T_e^*G^l, T_eG^l)\}.$$

These algebras are not stable by the Poisson brackets, but we still get an analoguous result. We denote by $m^l : T^*G \to T_e^*G^l$ the co-restriction of the momentum map to $T_e^*G^l$. We also denote by $i_{l+1} : T_eG^{l+1} \hookrightarrow T_eG^l$ the smooth inclusion of $T_eG^{l+1}$ in $T_eG^l$, and $i^*_{l+1} : T_e^*G^l \to T_e^*G^{l+1}$ its dual map (which is not necessarily injective since $T_eG^{l+1}$ is in general not dense in $T_eG^l$).

**Proposition 4.32** (Almost Lie-Poisson reduction)**.** *We get the following :*

1. $\{\mathcal{A}^{l+1}_{T^*G}, \mathcal{A}^{l+1}_{T^*G}\}_\omega \subset \mathcal{A}^l_{T^*G}$ *and* $\{\mathcal{A}_{T_e^*G^{l+1}}, \mathcal{A}_{T_e^*G^{l+1}}\}_+ \subset \mathcal{A}_{T_e^*G^l}$.
2. *The map* $m^l$ *is* $C^{l-1}$ *and* $(m^l)^*\mathcal{A}_{T_e^*G^l} \subset \mathcal{A}^{l-1}_{T^*G}$
3. *We get*

$$\forall h, f \in \mathcal{A}_{T_e^*G^l},\ \{h, f\}_+ \circ m^l = \{h \circ m^l, f \circ m^l\}_\omega. \tag{4.14}$$

**Remark 4.33.** *Note that in this case* $\mathcal{A}^{l+1}_{T^*G} \subset \mathcal{A}^{l}_{T^*G}$*, but not necessarily* $\mathcal{A}_{T_e^*G^{l+1}} \subset \mathcal{A}_{T_e^*G^l}$ *since the tangent space* $T_eG^l$ *is not dense in* $T_eG^{l+1}$ *in general. However any element* $h \in \mathcal{A}_{T_e^*G^{l+1}}$ *still defines a natural element* $h \circ i_l^* \in \mathcal{A}_{T_e^*G^l}$

*Proof.* The first inclusion $\{\mathcal{A}^{l+1}_{T^*G}, \mathcal{A}^{l+1}_{T^*G}\}_\omega \subset \mathcal{A}^{l}_{T^*G}$ is straightforward since for $F, H \in \mathcal{A}^{l+1}_{T^*G}$, then their bracket $\{F, H\}_\omega = \omega(X_F, X_H)$ is therefore $C^l$ and we get

$$d\{F, H\}_\omega = d\omega(X_F, X_H) = \omega([X_F, X_H], \cdot)$$

where $[X_F, X_H]$ is the usual Lie bracket of vector fields of $T^*G$ [2], and where the second equality follows from the fact that $i_{X_H}\omega$ is closed, implying $\mathcal{L}_{X_H}\omega = 0$ and from Cartan formula [76, Proposition 2.18]. We prove the second inclusion. Let $f, h \in \mathcal{A}_{T_e^*G^{l+1}}$. These two functions define functions in $C^\infty(T_e^*G^l)$ by $f \circ i_l^*$, $h \circ i_l^*$ that we also simply denote $h, f$. For $m \in T_e^*G^l$, we consider the second derivative $d^2f(m), d^2h(m) \in L^2(T_e^*G^{l+1}, T_e^*G^{l+1}; \mathbb{R})$ which are bilinear and symmetric. Since we have $dh(m), df(m) \in T_eG^{l+1}$, so that the second derivatives also induce linear maps

$$\widetilde{d^2h}(m), \widetilde{d^2f}(m) \in L(T_e^*G^{l+1}, T_eG^{l+1}) \to \mathbb{R}$$

so that, for any $\delta m, \delta m' \in T_e^*G^{l+1}$

$$(\delta m \mid \widetilde{d^2f}(m)(\delta m')) = d^2f(m)(\delta m, \delta m') = (\delta m' \mid \widetilde{d^2f}(m)(\delta m))$$

and similarly for $d^2h$. Moreover, the adjoint representation induces, for $v \in T_eG^{l+1}$ a dual map $\mathrm{ad}_v^* : T_e^*G^l \to T_e^*G^{l+1}$, so that the Hamiltonian vector fields $\hat{X}_f$ and $\hat{X}_h$ are in $C^\infty(T_e^*G^{l+1}, T_e^*G^l)$, and we get for $m \in T_e^*G^l$, we can define the bracket

$$k(m) = (m \mid [dh(m), df(m)]) .$$

Its derivative is given, for every $\delta m \in T_e^*G^l$, by

$$\begin{aligned}
dk(m)\delta m &= (\delta m \mid [df(m), dh(m)]) + \Big(m \mid [\widetilde{d^2f}(m)\delta m, dh(m)] + [df(m), \widetilde{d^2h}(m)\delta m]\Big) \\
&= (\delta m \mid [df(m), dh(m)]) + d^2f(m)(\delta m, \hat{X}_h(m)) - d^2h(m)(\delta m, \hat{X}_f(m)) \\
&= \Big(\delta m \mid [df(m), dh(m)] + \widetilde{d^2f}(m)(\hat{X}_h(m)) - \widetilde{d^2h}(m)(\hat{X}_f(m))\Big)
\end{aligned}$$

so that $dk(m) = [df(m), dh(m)] + d^2f(m)(\hat{X}_h(m)) - d^2h(m)(\hat{X}_f(m)) \in T_eG^l$.

We prove now the second point, i.e. the regularity of the map

$$m^l : \left\{ \begin{array}{ccc} T^*G & \longrightarrow & T_e^*G^l \\ (g, p) & \longmapsto & T_eR_g^*p \end{array} \right. .$$

Since, by (G.4), the map $T_eR : G \times T_eG^l \to TG$ is $C^l$, it thus induces a $C^{l-1}$ vector bundle morphism (cf. 3.21), so that the dual map $T_eR^* : G \to L(T^*G, T_e^*G^l)$ is also $C^{l-1}$. In particular the momentum $m^l$ is also $C^{l-1}$. Moreover, if $h \in \mathcal{A}_{T_e^*G^l}$, then the composition $H = h \circ m^l$ thus defines a $C^{l-1}$ map from $T^*G$ to $\mathbb{R}$. Let $(g, p) \in T^*G$, we compute in local charts adapted to the cotangent bundle $T^*G$ the partial derivative $\partial_pH$:

$$\begin{aligned}
\partial_pH(g, p)\delta p &= dh(m^l(g, p)) \cdot \partial_pm^l(g, p)\delta p \\
&= \big(T_eR_g^*\delta p \mid dh(m^l(g, p))\big) \\
&= \big(\delta p \mid T_eR_g dh\left(m^l(g, p)\right)\big)
\end{aligned}$$

and therefore $\partial_p H(g,p)\delta p = T_e R_g dh\left(m^l(g,p)\right) \in T_g G \subset T_g^{**}G$, i.e. $H$ admits a symplectic gradient, which proves that $H \in \mathcal{A}^{l-1}_{T^*G}$. Let us compute also the partial derivative $\partial_g H(g,p)$ in order to compute this symplectic gradient. Since $dh(m^l(g,p)) \in T_e G^l$, we get

$$\begin{aligned}\partial_g H(g,p) &= \big(p \mid \partial_g(T_e R_g)\delta g dh(m^l(g,p))\big)\\ &= \big(p \mid \partial_g\left(T_e R_g dh(m^l(g,p))\right)\delta g\big)\end{aligned}$$

Finally, this symplectic gradient is given by

$$\nabla^\omega H(g,p) = \Big(T_e R_g dh(m^l(g,p)), -\partial_g\left(T_e R_g dh(m^l(g,p))\right)^* p\Big)$$

Let now $F = f \circ m^l \in \mathcal{A}^{l-1}_{T^*G}$, with $f \in \mathcal{A}_{T_e^*G^l}$. We get

$$\begin{aligned}&\{F\circ m^l, H\circ m^l\}_\omega(g,p) = dF(g,p)(\nabla^\omega H(g,p))\\
&= \big(p \mid \partial_g(T_eR_g df(m^l(g,p))T_eR_g dh(m^l(g,p)) - \partial_g(T_eR_g dh(m^l(g,p))T_eR_g df(m^l(g,p))\big)\\
&= \big(p \mid T_eR_g\partial_g(T_eR_g df(m^l(g,p))|_{g=e} dh(m^l(g,p)) - T_eR_g\partial_g(T_eR_g dh(m^l(g,p))|_{g=e} df(m^l(g,p))\big)\\
&= \big(T_eR_g^* p \mid \partial_g(T_eR_g df(m^l(g,p))|_{g=e} dh(m^l(g,p)) - \partial_g(T_eR_g dh(m^l(g,p))|_{g=e} df(m^l(g,p))\big)\\
&= (m^l(g,p) \mid [df(m^l(g,p)), dh(m^l(g,p))]) = \{f,h\}_+ \circ m^l(g,p)\end{aligned}$$

where we used the fact that the maps

$$g' \mapsto T_e R_{g'} f(m^l(g,p)), T_e R_{g'} h(m^l(g,p))$$

are right-invariant and also that

$$\begin{aligned}&\big[df(m^l(g,p)), dh(m^l(g,p))\big] =\\ &\qquad \partial_{g'}(T_eR_{g'} df(m^l(g,p))|_{g'=e} dh(m^l(g,p)) - \partial_{g'}(T_eR_{g'} dh(m^l(g,p))|_{g'=e} df(m^l(g,p))\end{aligned}$$

(cf. 4.12). This proves the last point. □

We recall that the restriction on $T_eG^{l+1}$ of the adjoint mapping $\operatorname{ad} T_eG^{l+1} \times T_eG^{l+1} \to T_eG^l$ only co-restricts to $T_eG^l$, so that we lose a degree of regularity. For any $v \in T_eG^{l+1}$, it induces the dual morphism

$$\operatorname{ad}_v^* : T_e^*G^l \to T_e^*G^{l+1}$$

**Proposition 4.34** (Hamiltonian vector field of $\mathcal{A}_{T_e^*G^l}$)**.** *Let $f \in \mathcal{A}_{T_e^*G^{l+1}}$, and define the smooth vector field $\hat{X}_f \in C^\infty(T_e^*G^l, T_e^*G^{l+1})$ by*

$$\forall m \in T_e^*G^l,\ \hat{X}_f(m) = -\operatorname{ad}^*_{df\circ i^*_{l+1}(m)} m.$$

*Then for all $h \in \mathcal{A}_{T_e^*G^{l+1}}$, we have*

$$\forall m \in T_e^*G^l,\ \{h,f\}_+(m) = \mathcal{L}_{\hat{X}_f} h(m)$$

**Remark 4.35.** *Here the space $C^\infty(T_e^*G^l, T_e^*G^{l+1})$ is the natural space for Hamiltonian vector fields of Hamiltonian functions in $\mathcal{A}_{T_e^*G^{l+1}}$. Indeed, if $h \in \mathcal{A}_{T_e^*G^{l+1}}$, and $X \in C^\infty(T_e^*G^l, T_e^*G^{l+1})$, the derivative $\mathcal{L}_X h = dh(X)$ is well defined and in $\mathcal{A}_{T_e^*G^l}$ by composition :*

$$T_e^*G^l \xrightarrow{X} T_e^*G^{l+1} \xrightarrow{dh} T_eG^{l+1} \xhookrightarrow{i_{l+1}} T_eG^l$$

*Proof.* Define $\hat{X}_f \in C^\infty(T_e^*G^l, T_e^*G^{l+1})$ as in the proposition. Let $h \in \mathcal{A}_{T_e^*G^{l+1}}$, and $m \in T_e^*G^l$, we have

$$\begin{aligned}
\{h, f\}_+(m) &= (m \mid [dh(m), df(m)]) \\
&= -\Big(m \mid \mathrm{ad}_{df\circ i_{l+1}^*(m)}\, dh \circ i_{l+1}^*(m)\Big) \\
&= -\Big(\mathrm{ad}^*_{df\circ i_{l+1}^*(m)}\, m \mid dh \circ i_{l+1}^*(m)\Big) \\
&= \mathcal{L}_{\hat{X}_f} h(m)
\end{aligned}$$

which proves the proposition. $\square$

# Chapter 5

# Strong right invariant sub-Riemannian geometry

## Contents



In this chapter, we describe the main results from [82] and [81]. We follow the setting introduced in [5] and define strong right-invariant sub-Riemannian structures on half-Lie groups. If $G$ is a Banach half-Lie group, we consider a distribution $\Delta$ of subspaces of the tangent bundle, endowed with a metric $\langle\cdot,\cdot\rangle \in \Gamma(S_2^{++}\Delta)$. Such structures are called strong in the sense that the spaces $(\Delta_g, \langle\cdot,\cdot\rangle_g)$ become Hilbert spaces (in particular it a different approach from [48] where $\Delta$ is closed and admits a closed complement). We prove, given some hypotheses completeness of such structures, that is to say metric completeness, geodesic connectedness, and well-posedness of the (normal) geodesic equation. This extends results from [8], where authors prove completeness of such metrics on the particular group of Sobolev diffeomorphisms.

Following exposition of section 4.5, we derive a momentum formulation of the normal geodesic equations, and recover the sub-Riemannian Euler-Arnold-Poincaré equations [8]. This corresponds, in the case where the half-Lie group is the group of $C^k$-diffeomorphisms, to the EPDiff equations [75].

## 5.1 Right invariant sub-Riemannian geometry on Half-Lie groups

### 5.1.1 Definitions and basic properties

The LDDMM framework uses a space of velocity fields $V$ that is a Reproducing Kernel Hilbert Space (RKHS), continuously embedded in $C_0^{k+2}(\mathbb{R}^d, \mathbb{R}^d) = T_{\mathrm{id}} \operatorname{Diff}_{C_0^{k+2}}(\mathbb{R}^d)$. This induces a right-invariant sub-Riemannian metric on the group $\operatorname{Diff}_{C_0^k}(\mathbb{R}^d)$, as described in [8, 9, 6]. We use this example to define strong right-invariant sub-Riemannian metrics on $G$.

Let $\{G^k, k \geq 0\}$ be an admissible graded group structure 4.3, with $G = G^0$. Let $V$ be a Hilbert space continuously embedded in $T_e G^k$. We define, as in [6, 8], the sub-Riemannian structure $(G, G \times V, T_e R, \langle \cdot, \cdot \rangle_V)$:

- The space $G \times V$ defines a trivial vector bundle over $G$.
- The map
$$T_e R : \begin{array}{ccc} G \times V & \longrightarrow & TG \\ (g, u) & \longmapsto & T_e R_g(u) \end{array}$$
is a vector bundle morphism.
- The metric $\langle ., . \rangle_V$ denotes the scalar product inducing the Hilbert structure on $V$, and induces a metric on the bundle $G \times V$ that is independent of $G$:
$$\langle u, v \rangle_g = \langle u, v \rangle_V, \quad g \in G,\ u, v \in V$$

Note that by (G.3), the mapping $T_e R$ is $C^k$. By proposition 3.21 induced morphism $g \in G \mapsto T_e R_g \in L(V, TG)$ is only $C^{k-1}$, meaning $T_e R$ defines a $C^{k-1}$ bundle morphism. The sub-Riemannian structure is thus only $C^{k-1}$. Equivalently this also defines the *horizontal sub-Riemannian bundle* of the sub-Riemannian structure by

$$\Delta_g \coloneqq T_e R_g(V) \subset T_g G$$

The metric $\langle ., . \rangle_V$ induces a continuous metric on the horizontal bundle

$$\langle \delta g, \delta g' \rangle_g = \langle T_e R_{g^{-1}}(\delta g), T_e R_{g^{-1}} \delta g' \rangle_V, \quad g \in G,\ \delta g, \delta g' \in T_g G$$

This metric is also right-invariant, meaning that for any $g, g' \in G$, $\delta g, \delta g' \in T_g G$, we get

$$\langle T_g R_{g'} \delta g, T_g R_{g'} \delta g' \rangle_{gg'} = \langle \delta g, \delta g' \rangle_g.$$

**Remark 5.1.** *This explains the terminology of sub-Riemannian structure. Contrary to usual Riemannian geometry, the metric $\langle \cdot, \cdot \rangle$ here is only defined on a subspace of the tangent bundle $TG$, meaning only certain directions are allowed. Note that in our case, most of the times the subspace $\Delta_g$ will be dense in $T_g G$, meaning that the structure is not really restrictive. We also refer to [48], where the authors introduce a metric on subbundle that admits a closed complement in the tangent space.*

### 5.1.2 Horizontal curves and distance

We now recall some vocabulary from sub-Riemannian geometry and adapt it to the sub-Riemannian structures we defined in the previous section. Let $I$ denote the interval $[0, 1]$. We start by defining the notion of horizontal curve and horizontal system.

**Definition 5.2** (Horizontal curves in $G$)**.** An absolutely continuous curve $g : I \to G$ is said to be *horizontal* if there exists a continuous lift $t \mapsto u(t) \in V$ such that

$$\forall t \in I,\ \dot{g}(t) = T_e R_{g(t)} u(t)$$

We call an *horizontal system* such a couple $(g, u) \in AC_{L^1}(I, G) \times L^1(I, V)$.

**Remark 5.3.** *The lift of the absolutely continuous curve $g : I \to G$ is unique, since the morphism $T_e R_g : V \to T_g G$ is injective.*

We denote by $\mathrm{Hor}(I)$ the space of horizontal systems,

$$\mathrm{Hor}(I) = \{(g, u) \in AC_{L^1}(I, G) \times L^1(I, V) \mid \dot{g}(t) = T_e R_{g(t)} u(t)\}.$$

The *length* and *energy* of a curve $(g, u) \in \mathrm{AC}_{L^1}(I, G) \times L^1(I, V)$ are given by

$$L(g, u) = \int_I |u(t)|_V dt \qquad \text{and} \qquad E(g, u) = \frac{1}{2} \int_I |u(t)|_V^2 dt.$$

Hence, $G$ becomes a metric space with the following *sub-Riemannian distance*

$$d_V(g_0, g_1) = \inf\{L(g, u),\ (g, u) \text{ is a } L^1 \text{ horizontal system joining } g_0 \text{ to } g_1\}.$$

Properties of the sub-Riemannian distance are summed up in the following proposition.

**Proposition 5.4** (Sub-riemannian distance)**.** *The sub-Riemannian distance $d_V$ is a true right-invariant distance, and the topology induced by the sub-Riemannian distance is coarser than the intrinsic topology on $G$. Furthermore the distance $d_V$ is also equal to the infimum of the energy on horizontal $L^2$ systems, i.e. for $g_0, g_1 \in G$:*

$$d_V(g_0, g_1) = \inf\{\sqrt{2E(g, u)},\ (g, u) \textit{is a horizontal system joining } g \textit{ to } g'\}.$$

*Proof.* The proof is mostly contained in [7], we include the proof here for sake of completeness. Let's prove first that the application $d_V$ separates points. Let $g_0,\ g_1 \in G$ be distinct, let $U \subset G \to \mathbb{B}$ be a chart of $G$ around $g_0$ with $(\mathbb{B}, |\cdot|_{\mathbb{B}})$ Banach space, and $\epsilon > 0$ such that the open ball $B_{\mathbb{B}}(g_0, \epsilon)$ does not contain $g_1$. Since the map $T_e R : U \to L(V, TU) \simeq L(V, \mathbb{B})$ is continuous, it is also locally bounded, and there exists an open ball $B' = B_{\mathbb{B}}(g_0, \epsilon') \subset B_{\mathbb{B}}(g_0, \epsilon)$, and $a > 0$ such that for all $q \in B'$, we have $\|T_e R_g\|_{L(V,\mathbb{B})} \leq a$, i.e. for all $q \in B', u \in V$

$$|T_e R_g(u)|_{\mathbb{B}} \leq a|u|_V$$

Now let $(g, u) : I \to G \times V$ horizontal system such that $g(0) = g_0$, and $g(1) = g_1$. As $g$ is continuous, there exists $t_0 \in I$ such that $|g(t_0) - g|_{\mathbb{B}} = \epsilon'$ and $g(t) \in B'$ for $t \leq t_0$. We get :

$$\begin{aligned} L(g, u) &\geq L(g_{|[0,t_0]}, u_{|[0,t_0]}) \geq \int_0^{t_0} |u(t)|_V dt \\ &\geq \frac{1}{a} \int_0^{t_0} |T_e R_g(u)|_{\mathbb{B}} dt = \frac{1}{a} \int_0^{t_0} |\dot{g}(t)|_{\mathbb{B}} dt \geq \frac{1}{a} |g(t_0)|_{\mathbb{B}} = \epsilon'/a \end{aligned}$$

Therefore we conclude that $d_V(g_0, g_1) > 0$. Right-invariance of the distance comes from the fact the metric is right-invariant. Indeed, if $(g, u)$ is a $L^1$ horizontal system joining $g_0$ to $g_1$, the curve $(gg_0^{-1}, u)$ is a $L^1$ horizontal system joining $e$ to $g_1 g_0^{-1}$ and we get

$$L(gg_0^{-1}, u) = L(g, u)$$

so that $d_V(g_0, g_1) = d_V(e, g_1 g_0^{-1})$.

To prove the last point, we first see by Cauchy-Schwarz inequality that for $(g, u)$ a horizontal system, we have :

$$L(g, u) \leq \sqrt{2E(g, u)}.$$

with equality if and only if $(g, u)$ is constant speed (i.e. if $|u|_V$ is constant). Now for the reverse inequality, we approximate horizontal systems by constant speed reparametrization and show the length stay close. Indeed, let $(g, u)$ horizontal system with $u \in L^1(I, V)$. Then for $\epsilon > 0$, we can consider the increasing absolutely continuous bijection $s_\epsilon(t) = \left(\int_0^t |u_t|_V + \epsilon t\right) / (|u|_{L^1} + \epsilon)$, with inverse $s \to t_\epsilon(s)$ such that $\dot{t}_\epsilon(s) = (|u|_{L^1} + \epsilon)/(|u_{t_\epsilon(s)}|_V + \epsilon)$. Now we can define $\tilde{u}_s = \dot{t}_\epsilon(s) u_{t(s)}$ and see that $|\tilde{u}_s|_V \leq |u|_{L^1} + \epsilon$. Therefore $\tilde{u}$ is in $L^\infty(I, V) \subset L^2(I, V)$, and we also see that if $\tilde{g}_s = g_{t_\epsilon(s)}$, then :

$$\dot{\tilde{g}}_s = T_e R_{\tilde{g}_s} \tilde{u}_s$$

so that $(\tilde{g}, \tilde{u})$ is a $AC_{L^2}$ horizontal system with endpoints $g_0$ and $g_1$. We finally have that

$$|\tilde{u}|_{L^2} \leq |u|_{L^1} + \epsilon$$

which concludes the proof. $\square$

#### 5.1.2.1 Geodesics and completeness

We finish the section with some geodesic and metric completeness result. We first recall the definition of geodesics associated with the distance $d_V$ :

**Definition 5.5** (Geodesics in $G$)**.** Let $(g, u) \in AC_{L^2}(I, G) \times L^2(I, V)$ an horizontal system. Then

- We say that the curve $(g, u)$ is a *geodesic* if it minimizes locally the length, meaning for every $t_0 \in I$, and $t_1$ close enough to $t_0$ :

$$L((q, u)_{|[t_0, t_1]}) = d_V(q(t_0), q(t_1))\,.$$

- The curve $(g, u)$ is a *minimizing geodesic* if its total length is equal to the distance between the endpoints.

**Remark 5.6.** *We already saw in proof of Proposition 5.4 that if $(g, u)$ is a horizontal system that minimizes the energy, then it's immediately a minimizing geodesic, and is also parametrized with constant speed. Conversely, if $(g, u)$ is a minimizing geodesic parametrized with constant speed, then $(g, u)$ also minimizes the energy, and we have*

$$L(q, u) = \sqrt{2E(q, u)}\,.$$

Suppose any curve in $L^2(I,V)$ can be integrated in $G$ (for example if $V$ is included in $T_eG^1$ by proposition 4.18, or if $G$ is $L^2$-regular). Let us define the endpoint mapping

$$\text{End} : \begin{cases} L^2(I,V) & \longrightarrow & G \\ u & \longmapsto & \text{Evol}_G(u)(1) \end{cases}$$

We will need another assumption on the action of the groups $\{G^k, k \geq 0\}$ to prove that $(G, d_V)$ is a geodesic metric space, i.e. that we can join any two points of $G$ by a minimizing geodesic :

**(G.6)** The endpoint mapping End : $L^2(I,V) \to G$ is weakly-continuous where $G$ is equipped with some Hausdorff topology.

**Theorem 5.7** (Completeness of the metric)**.** *We get the following*

1. *The space $G$ with distance $d_V$ is metrically complete.*
2. *Moreover, if (G.6) is satisfied, then $G$ is a geodesic metric space, meaning for $g_0, g_1 \in G$ such that $d_V(g_0, g_1) < \infty$, there exists a minimizing geodesic connecting $g_0$ and $g_1$.*

*Proof.* We follow the proof from [95] in our more general context. We prove first $(G, d_V)$ is complete metric space. We consider a Cauchy sequence $(g_n)_n$ in $G$, and we can suppose $\sum_n d_V(g_n, g_{n+1}) < \infty$. Thus there exists $u_n \in L^1(I,V)$, such that $g_{n+1} = \text{Evol}_G(u_n)g_n$ and such that $|u_n|_{L^1(I,V)} < 2d_V(g_n, g_{n+1})$. We define a new sequence $w_n$ in $L^1(I,V)$ such that

$$w_n = \begin{cases} 2^{k+1}u_k(2^{k+1}(t-1)+2) & \text{for } t \in [\frac{2^k-1}{2^k}, \frac{2^{k+1}-1}{2^{k+1}}], k \in \{0,1,..,n\} \\ 0 \text{ for } t \in [\frac{2^{n+1}-1}{2^{n+1}}, 1] \end{cases}$$

Intuitively, we just concatenate the paths $u_n$, so that $g_{n+1} = \text{End}(w_n)$, and $|w_n|_{L^1(I,V)} = \sum_{k \leq n} |u_k|_{L^1(I,V)} < \infty$ and $|w_{n+1} - w_n| = |u_{n+1}|_{L^1(I,V)}$. Therefore $(w_n)$ is a Cauchy sequence in the Banach space $L^1(I,V)$, and thus converges to $w_\infty \in L^1(I,V)$. We denote $g_\infty = \text{End}(w_\infty)g_0 \in G$. We also see that $g_\infty = \text{End}(w_\infty - w_{n-1})g_n$, and thus

$$d_V(g_\infty, g_n) \leq |w_\infty - w_{n-1}| \to 0$$

Therefore $(G, d_V)$ is a complete metric space.

Now we prove the existence of geodesic between points. By right-invariance of the distance, it suffices to prove that if $g \in G$ is such that $d_V(e,g) < \infty$, there exists a geodesic from $e$ to $g$. Let $(g_n, u_n)$ a minimizing sequence for the energy, with $u_n \in L^2(I,V)$. Since $(u_n)$ is bounded in $L^2$, we can suppose, up to a subsequence, that $(u_n)$ converges weakly towards $u_\infty \in L^2(I,V)$. We define by $g_\infty = \text{Evol}_G(u_\infty)$ the horizontal curve such that $\dot{g}_\infty = T_eR_{g_\infty}u_\infty$. Since the endpoint mapping End is weakly continuous, and $g_n(1) = g$ for all $n$, then we also have $g_\infty = g$. Finally, since $|u_\infty|_{L^2} \leq \liminf |u_n|_{L^2} = d_V(e,g)$, we get the result. □

### 5.1.3 Sub-Riemannian geodesics and critical points of the energy

The aim of this section is to characterise geodesics and critical points of the energy in a sub-Riemannian setting, as in [8]. We suppose in this part that $V \hookrightarrow T_eG^2$. We start with a little discussion on the different types of sub-Riemannian geodesics as well as on the differences with the Riemannian case and the sub-Riemannian finite-dimensional case (as described in [73]). This was described for the infinite-dimensional case in [8, 7]. Since $V \subset T_e G^2$ and by hypothesis (G.4), and since the evaluation $g \in AC_{L^2}(I, G) \mapsto g(1) \in G$ is smooth by proposition 4.17, the endpoint mapping is thus $C^1$ by composition. We define the space of horizontal systems with endpoints $e$ and $g_1$ by :

$$\mathrm{Hor}_{g_1}(I) = \mathrm{End}^{-1}(\{g_1\}).$$

The main difficulties will come from the fact that the endpoint mapping End is not a submersion and therefore the space of horizontal curves with fixed endpoints $\mathrm{Hor}_{g_1}(I)$ might not be a manifold. More precisely, let $g_1 \in G$, and let $(g_{opt}, u_{opt}) \in AC_{L^2}(I, G) \times L^2(I, V)$ be a minimizing curve for the energy, such that $g_{opt}(1) = g_1$. Then, the mapping $(E, \mathrm{End}) : u \mapsto (E(u), \mathrm{End}(u))$ cannot be a submersion at $u_{opt}$; otherwise the map $(E, \mathrm{End})$ would be locally surjective from a neighborhood of $u_{opt}$ to a neighborhood of $(E(u_{opt}), \mathrm{End}(u_{opt}))$ [61, Proposition 2.2]. This leads to two different cases:

(i) $(dE(u_{opt}), d\,\mathrm{End}(u_{opt}))$ has closed range.

(ii) $(dE(u_{opt}), d\,\mathrm{End}(u_{opt}))$ has dense range in $\mathbb{R} \times T_{g_1}G$.

By the closed range theorem, case (i) is equivalent to the adjoint mapping

$$(dE(u_{opt}), d\,\mathrm{End}(u_{opt}))^* : \mathbb{R} \times T_{g_1}^* G \to L^2(I, V)^*$$

being not injective, meaning there exist non-zero Lagrange multipliers $(\lambda, p) \in \mathbb{R} \times T_{g_1}G$ such that

$$\lambda dE(u_{opt}) + (d\,\mathrm{End}(u_{opt}))^* p = 0 \tag{5.1}$$

which is the classical case of finite dimension. It separates into two subcases: horizontal curves that satisfy (5.1) with $\lambda \neq 0$ (in this case we can simply choose $\lambda = -1$) are called *normal sub-Riemannian geodesics*, or simply *normal geodesics*. In particular, minimizing geodesics that are regular curves for the endpoint map are normal geodesics. We will derive those geodesic equations in the next section. If $\lambda = 0$, the geodesics are *singular curves* for the endpoint map. Such geodesics can also be characterized by an abnormal geodesic equation [73]. In case (ii), nontrivial Lagrange multipliers do not exist. Such geodesics are called *elusive geodesics* [7], and can occur only in infinite dimension. In the next section, we will only focus on the normal geodesics.

**Definition 5.8** (Sub-Riemannian normal geodesic)**.** Let $(g, u) \in \mathrm{Hor}(I)$ an horizontal system. We say that $(g, u)$ is a *sub-Riemannian normal geodesic* (or just *normal geodesic*) if there exists Lagrange multipliers $(\lambda, p) \in \mathbb{R} \times T_{g(1)}G^*$, with $\lambda$ not equal to zero such that

$$\lambda dE(u)\delta u + (p \mid d\,\mathrm{End}(u)\delta u) = 0 \tag{5.2}$$

**Remark 5.9.** *Normal geodesics are in particular geodesics in the sense of definition 5.5, i.e. they minimize locally the energy [7, Theorem 7]. They correspond to Riemannian geodesics in classic riemannian geometry.*

## 5.2 Normal geodesics and Hamiltonian equation

In this section, we study the normal geodesic equation associated with a strong right-invariant sub-Riemannian metric, and we show that this equation is equivalent to a reduced geodesic equation in the dual of the Lie algebra (the Euler-Arnold equations). We recall that $V \hookrightarrow T_eG^k$ is a Hilbert space, and we suppose $k \geq 1$.

### 5.2.1 Hamiltonian equations for the normal geodesics

We define the normal pre-Hamiltonian $\mathcal{H} : T^*G \times V \to \mathbb{R}$ by

$$\mathcal{H}(g, p, u) = (p \mid T_eR_g(u)) - \frac{1}{2}|u|_V^2$$

By differentiability of the composition law in the group $G$, i.e. (G.3), the Hamiltonian $\mathcal{H}$ is $C^1$. Suppose $u \in V$ is fixed, and let us compute the derivatives. In local coordinates adapted to the cotangent bundle $T^*G$, we get for any $(g, p) \in T^*G$,

$$\partial_p\mathcal{H}(g, p, u)\delta p = (\delta p \mid T_eR_gu).$$

In particular, the partial derivative $\partial_p\mathcal{H}(g, p, u) = T_eR_gu$ so that the Hamiltonian $\mathcal{H}$ admits a partial (in the sense that $u$ is fixed) symplectic gradient given by

$$\nabla^\omega\mathcal{H}(g, p, u) = (T_eR_gu, -\partial_g(T_eR_g(u))^*p)$$

**Proposition 5.10** (Normal Hamiltonian equations)**.** *Suppose $V \subset T_eG^k$, with $k \geq 1$. Let $g \in \mathrm{AC}_{L^2}(I, G)$ be an horizontal curve. Then, $g_t$ is a normal geodesic if and only if it is the projection onto G of a curve $(g, p) \in \mathrm{AC}_{L^2}(I, T^*G)$ satisfying the Hamiltonian equations:*

$$\begin{cases} (\dot{g}_t, \dot{p}_t) = \nabla^\omega\mathcal{H}(g_t, p_t, u_t). \\ \partial_u\mathcal{H}(g_t, p_t, u_t) = 0 \end{cases} \tag{5.3}$$

**Remark 5.11.** *The last condition of equation* (5.3)

$$\partial_u\mathcal{H}(g, p, u) = 0 = (p \mid T_eR_g(.)) - \langle u, \cdot\rangle_V$$

*has a unique solution given by $u(g, p) = K_VT_eR_g^*p$, where $K_V : V^* \to V$ is the inverse of the Riesz canonical isometry. Since $\mathcal{H}$ is strictly concave in $u$, we also have that $u(g, p) = \max_{u\in V} \mathcal{H}(g, p, u)$. We will use this equality to prove the Hamiltonian flow is global and unique in the next section.*

*Proof.* Let $u \in L^2(I, V)$, $g^u(t) = \mathrm{Evol}_G(u)(t)$, and let $p_1 \in T^*_{g_1}G$. Define $p^u \in AC_{L^1}(I, T^*G)$ the solution of the following linear Cauchy problem :

$$\begin{cases} \dot{p}^u(t) = -\partial_g\mathcal{H}(g^u(t), p^u(t), u(t)) = -\left(\partial_g(T_eR_gu(t))|_{g=g^u(t)}\right)^*p^u(t) \\ p^u(1) = p_1 \end{cases} \tag{5.4}$$

we introduce the linear map $F : L^2(I, V) \to \mathbb{R}$, defined, for all $\delta u \in L^2(I, V)$, by :

$$F(\delta u) = dE(u)\delta u - (p_1 \mid d\,\mathrm{End}(u)\delta)$$

and we prove that for all $\delta u \in L^2(I, V)$,

$$F(\delta u) = -\int_I \partial_u\mathcal{H}(g^u(t), p^u(t), u(t))\delta u dt$$

Let $\delta u \in L^2(I, V)$, we have

$$\begin{aligned} F(\delta u) &= dE(u)\delta u - (p_1 \mid d\,\mathrm{End}(u)\delta u) \\ &= \int_I \langle u, \delta u\rangle_V dt - (p^u(1) \mid d\,\mathrm{End}(u)\delta u) \end{aligned}$$

We recall that $\delta g = T_u\,\mathrm{Evol}_G\,\delta u$ satisfies the linear Cauchy problem (4.8)

$$\delta\dot{g}(t) = \partial_g\left(T_e R_g u(t)\right)|_{g=g^u(t)}\delta q(t) + T_e R_{g^u(t)}(\delta u(t)), \quad \delta g(0) = 0$$

Moreover we also get that

$$d\,\mathrm{End}(u)\delta u = \delta g(1).$$

Now, as $t \mapsto p^u(t)$ is also solution of the linear Cauchy equation (5.4) and using integration by part, we find that

$$\begin{aligned} (p^u(1) \mid d\,\mathrm{End}(u)\delta u) &= (p^u(1) \mid \delta g(1)) \\ &= (p^u(0) \mid \delta g(0)) + \int_I \left(\dot{p^u}(t) \mid \delta g(t)\right) + (p^u(t) \mid \delta\dot{g}(t))\,dt \\ &= \int_I -\left( \left(\partial_g(T_e R_g u(t))|_{g=g^u(t)}\right)^* p^u(t) \mid \delta g(t)\right) \\ &\quad + \left(p^u(t) \mid \partial_g\left(T_e R_g u(t)\right)|_{g=g^u(t)}\delta g(t) + T_e R_{g^u(t)}(\delta u(t))\right) dt \\ &= \int_I \left(p^u(t) \mid T_e R_{g^u(t)}(\delta u(t))\right) dt \end{aligned}$$

Finally this gives us

$$\begin{aligned} F(\delta u) &= \int_I \langle u, \delta u\rangle_V dt - \int_I \left(p^u(t) \mid T_e R_{g^u(t)}(\delta u(t))\right) dt \\ &= -\int_I \partial_u \mathcal{H}(g^u(t), p^u(t), u(t))\delta u dt \end{aligned}$$

Therefore we have the following equivalence :

$$dE(u) = d\,\mathrm{End}(u)^* p_1 \iff \forall t,\ \partial_u \mathcal{H}(g^u(t), p^u(t), u(t)) = 0 \tag{5.5}$$

which concludes the proof by definition of the normal geodesics. □

### 5.2.2 Global existence of the Hamiltonian flow

We denote by $K : V^* \to V$ the inverse of the Riesz canonical isometry. For $l \leq k$, let $i_l : V \to T_e G^l$ denote the smooth inclusion, and $i_l^* : T_e^* G^l \to V^*$ be its adjoint. The mapping $i_l^*$ is not necessarily injective unless $V$ is dense in $T_e G^l$ (even though it is usually dense in the case in the context of LDDMM). We also denote by $K^l : T_e^* G^l \to V$ the composition $K^l = K \circ i_l^*$. This mapping induces a vector bundle morphism, which we also denote by $K^l : T^* G^l \to TG^l$ and which is defined as

$$K^l : \left\{ \begin{array}{ccc} T^* G^l & \longrightarrow & TG^l \\ (g, p) & \longmapsto & K_g^l p, \end{array} \right.$$

where $K^l_g = T_eR_g \circ K^l \circ T_eR^*_g$. This mapping defines the cometric for the sub-Riemannian structure induced by $V$ on $G^l$, for $l \leq k$. We now define the *normal sub-Riemannian Hamiltonian* as

$$H : \left\{ \begin{array}{ccl} T^*G & \longrightarrow & \mathbb{R} \\ (g,p) & \longmapsto & \frac{1}{2}\big(p \mid K^0_g p\big) = \frac{1}{2}|K^0 T_e R^*_g p|^2_V \end{array} \right.$$

This Hamiltonian can be derived from the pre-Hamiltonian $\mathcal{H} : T^*G \times V \to \mathbb{R}$ by the problem

$$H(g,p) = \sup_{u \in V} \mathcal{H}(g,p,u)$$

whose supremum is attained for $u(g,p) = K^0 T_e R^*_g p$.

This normal Hamiltonian is $C^1$ whenever $k \geq 2$, i.e. if $V$ is embedded in $T_eG^2$. Let us compute its derivatives. We work in local coordinates adapted to the cotangent bundle $T^*G$, and we compute the partial derivative $\partial_p H(g,p) \in T^{**}G$:

$$\partial_p H(g,p)\delta p = \big(\delta p \mid K^0_g p\big),$$

so that we actually have $\partial_p H(g,p) = K^0_g p \in TG \subset T^{**}G$. Therefore, we can define the symplectic gradient of the normal Hamiltonian with respect to the canonical weak symplectic form :

$$\nabla^\omega H(g,p) = (\partial_p H(g,p), -\partial_g H(g,p)) = \big(K^0_g p, -\big(\partial_g K^0_g p\big)^* p\big).$$

In particular, note that $\nabla^\omega H(g,p) = \nabla^\omega \mathcal{H}(g,p,u(g,p))$, where $u(g,p) = K^0_g p \in TG$, which means equations (5.3) can be simply re-written as $(\dot g, \dot p) = \nabla^\omega H(g,p)$. We prove next that the Hamiltonian flow exists, is unique and is global.

**Proposition 5.12** (Global existence and uniqueness of the Hamiltonian flow)**.** *Suppose $V \hookrightarrow T_eG^2$. Then, for every $p \in T^*_eG$, there exists a unique global curve $(g_t, p_t) \in$ $\mathrm{AC}_{L^2}(I, T^*G)$ satisfying the normal geodesic equation.*

$$(\dot g_t, \dot p_t) = \nabla^\omega H(g_t, p_t), \quad (g_0, p_0) = (e, p) \tag{5.6}$$

*Moreover the curves $t \mapsto (g_t, p_t) \in T^*G$ and $t \mapsto g_t \in G^1$ are actually $C^1$.*

*Proof. Step 1 : Local existence and uniqueness :* As the restriction $T_eR : G \times V \to TG$ is a $C^2$ mapping by (G.4), the induced map $g \in G \mapsto T_eR_g|_V \in L(V, TG)$ is $C^1$ with locally lipschitz derivative (proposition 3.21), and therefore $(g,p) \mapsto T_eR^*_g p|_V \in V^*$ is $C^1$ with locally lipschitz derivative. Finally the mapping

$$u : \left\{ \begin{array}{ccl} T^*G & \longrightarrow & V \\ (g,p) & \longmapsto & K^0_g p = K^0 T_e R^*_g p \end{array} \right.$$

is $C^1$ with locally lipschitz derivative. Thus, since for any $(g,p) \in T^*G$, $H(g,p) = \frac{1}{2}|u(g,p)|^2_V$, the Hamiltonian $H$ is $C^1$ with locally lipschitz derivative and its symplectic gradient is locally lipschitz. By Picard-Lindelof theorem, we can therefore integrate the equation:

$$(\dot g, \dot p) = \nabla^\omega H(q,p)$$

and get for any initial data $(e, p_0) \in T^*_eG$ a unique Hamiltonian $C^1$ flow $t \mapsto (g_t, p_t)$.

*Step 2 : Regularity of the eulerian derivative $u_t$ and of $g_t$ :* We denote by $u_t = u(g_t, p_t) \in V$ the Eulerian derivative, so that for all $t \in J$, $H(g_t, p_t) = \frac{1}{2}|u_t|_V^2$ and

$$\dot{g}_t = T_e R_{g_t} u_t = u_t \cdot g_t$$

Moreover, since $u_t = u(g_t, p_t)$, and the curve $(g_t, p_t) : J \to T^*G$ is $C^1$, then the curve $t \mapsto u_t \in V \subset T_e G^2$ is also $C^1$. In particular, by proposition 4.7, $u_t$ can be integrated in a $C^1$ curve in $G^1$. This means that the curve $t \mapsto g_t$ is in $G^1$ and is $C^1$ from $J$ to $G^1$.

*Step 3 : The Hamiltonian flow is defined globally :* We can now prove that $J = \mathbb{R}$. To prove that $(g_t, p_t)$ is defined on $\mathbb{R}$, we first show that $|u_t|$ is constant. For $t \in J$, we have

$$\begin{aligned}
\frac{d}{dt} H(g_t, p_t) &= \partial_q H(g_t, p_t)(\dot{g}_t) + \partial_p H(g_t, p_t)(\dot{p}_t) \\
&= \partial_q H(g_t, p_t)(\partial_p H(g_t, p_t)) + \partial_p H(g_t, p_t)(-\partial_q H(g_t, p_t)) \\
&= 0.
\end{aligned}$$

But we also have $H(g_t, p_t) = \mathcal{H}(g_t, p_t, u_t) = \frac{1}{2}|u_t|_V^2$, therefore $u_t$ is constant.

Let now $b = \sup J$, and suppose $b < +\infty$. We are going to prove that in that case, the Hamiltonian flow $(g_t, p_t)$ converges and thus we get a contradiction. We recall that by (G.4), $(g, u) \in G^1 \times T_e G^1 \mapsto T_e R_g u \in TG^1$ is continuous. Then by [79, Theorem 5.3], the mapping $g \in G^1 \to T_e R_g \in L(T_e G^1, TG^1)$ is also locally bounded. Let $(U, \phi)$ a local chart of $G^1$ around $e$, and so that there exists $K > 0$, such that for all $g \in U$

$$\|d_g \phi \circ T_e R_g\| < K\,.$$

Such an open set U always exists since, if we identify $T_e G^1$ and the Banach space $\mathbb{B}^1$, the map $g \in G^1 \mapsto d_g \phi \circ T_e R_g \in L(T_e G^1, T_e G^1)$ is locally bounded. Let $t_0 \in J$, and by translating the curve $g_t$ we define $\tilde{g}_t \in G^1$ :

$$\tilde{g}_t = g_t g_{t_0}^{-1} = R_{g_{t_0}^{-1}} g_t\,.$$

The curve $\tilde{g}_t$ verifies the Cauchy problem :

$$\left\{ \begin{array}{l} \dot{\tilde{g}}_t = u_t \cdot \tilde{g}_t = T_e R_{\tilde{g}_t}(u_t)\,, \\ \tilde{g}_0 = e\,. \end{array} \right. \tag{5.7}$$

On the open subset $\tilde{J} = \{t \in J, \tilde{g}_t \in U\}$, we can therefore look at the curve $\alpha_t = \phi(\tilde{g}_t)$ on the local chart $U$, whose derivative is given by $\dot{\alpha}_t = d_{\tilde{g}_t} \phi \circ T_e R_{\tilde{g}_t}(u_t)$. Therefore, for all $t \in \tilde{J}$,

$$\begin{aligned}
|\dot{\alpha}_t|_{T_e G^1} &= |d_{\tilde{g}_t} \phi \circ T_e R_{\tilde{g}_t}(u_t)|_{T_e G^1} \\
&\leq K |u_t|_{T_e G^1} \\
&\leq K' |u_t|_V \quad \text{as } V \hookrightarrow T_e G^1
\end{aligned}$$

and as $u_t$ is bounded, by [61, prop 1.1, p.68], there exists $a, \epsilon > 0$ independent of $t_0$ so that the flow $\alpha_t$ can be defined on $]t_0 - \epsilon, t_0 + \epsilon[ \cap J$ and $\alpha_t \in \phi(U)$. Therefore, we can take $t_0 \in ]b - \epsilon, b[$, the curve $\alpha_t$ will also be defined on $]b - \epsilon, b[$, is Lipschitz and hence converges towards $\alpha_b \in \phi(U)$. We also have that $p_t$ is solution of the linear differential equation :

$$\dot{p}_t = -\partial_q H(g_t, p_t) = -\left(\partial_g (T_e R_g(u_t))|_{g=g_t}\right)^* p_t$$

The mapping $(g,u) \mapsto (\partial_g(T_eR_g(u)))^*$ is continuous and linear with regards to $u$, therefore there exists $L > 0$ and $\delta > 0$ such that in a local chart around $g_b$, and for $|u| < \delta$, we have :

$$\|(\partial_g(T_eR_g(u_t)))^*\|_{L(\mathbb{B}^*,\mathbb{B}^*)} \leq L$$

where $\mathbb{B}$ is a Banach model space for $G$, and where we identify $\partial_g(T_eR_g(u_t))^*$ with its induced mapping in the chart around $g_b$. Therefore, for $t$ close to $b$, we also have :

$$|\left(\partial_g(T_eR_g(u_t))|_{g=g_t}\right)^*|_{L(\mathbb{B}^*,\mathbb{B}^*)} \leq L|u_t|/\delta = L|u_0|/\delta$$

Therefore by Gronwall lemma, $p$ is bounded when $t \in ]b-\epsilon', b[$ with $\epsilon' > 0$, and thus $\dot{p}$ is also bounded. This implies again that $p_t$ converges in $T^*G$ towards $p_b$.

Therefore the Hamiltonian flow $(g_t, p_t)$ converges when $t \to b$, and then $b = +\infty$. A similar proof would show that we also have $\inf J = -\infty$, hence $J = \mathbb{R}$ and this concludes the proof. □

### 5.2.3 Sub-Riemannian Euler-Arnold equations

Following section 4.5.2, since the normal Hamiltonian is right-invariant, we can derive a momentum formulation of the normal geodesic equations. For any $(g,p) \in T^*G$, we have

$$H(g,p) = h_e \circ m(g,p)$$

where $h_e(\nu) = H(e,\nu) = \frac{1}{2}|\nu|^2_{V^*}$, for $\nu \in T_e^*G$, and $m : T^*G \to T_e^*G^2$ is the momentum map defined as

$$\forall u \in T_eG^2,\ (m(g,p) \mid u) = (p \mid T_eR_g(u))\,.$$

This momentum map and this new Hamiltonian give rise to another dynamic, which has been particularly studied for Riemannian geometry in infinite dimension [10, 32, 16] and for sub-Riemannian metrics on diffeomorphism groups [8].

**Theorem 5.13** (Sub-Riemannian Euler-Arnold equation)**.** *Suppose $V \hookrightarrow T_eG^2$. Let $g \in \mathrm{AC}_{L^2}(I,G)$ be a horizontal curve, with $u_t = \dot{g}_t g_t^{-1} = (T_eR_{g_t})^{-1}\dot{g}_t$. Then, $g_t$ is a normal geodesic if and only if there exists a momentum map $m_t \in C(I, T_e^*G^1) \cap C^1(I, T_e^*G^2)$, such that $K^1 m_t = u_t$ and that satisfies the equation (in $T_e^*G^1$) :*

$$\dot{m}_t + \mathrm{ad}^*_{u_t} m_t = 0. \tag{5.8}$$

*In such case, the covector $p(t) = (T_eR^*_{g_t})^{-1}m(t)$ defines a curve $(g,p) \in \mathrm{AC}_{L^2}(I, T^*G)$ that satisfies the normal Hamiltonian equation* (5.3)

**Remark 5.14.** *In this case, using the notations of 4.5.2, we see that $H \in \mathcal{A}^2_{T^*G}$ and $h_e \in \mathcal{A}_{T_e^*G^2}$. Since, for $\nu \in T_e^*G^2$, $dh_e(m_t) = K^2 m_t = K_V i_V^{2*}\nu$.*

*Proof.* Let $(g_t, p_t) \in C^1(I, T^*G)$ a solution of the Hamiltonian equations (5.6), with $p_0 \in T_eG^*$. We define the momentum $m_t = T_eR^*_{g_t}p_t \in T_e^*G$, so that for $v \in T_eG$,

$$(m_t \mid v) = (p_t \mid T_eR_{g_t}v).$$

With a slight abuse of notation, we can consider that $m_t$ is also in $T_e^*G^1$ and $T_e^*G^2$. Let $l \in \{1,2\}$. As the restriction $T_eR : G \times T_eG^l \to TG$ is a $C^l$ mapping (G.4), the induced map $g \in G \mapsto T_eR_g \in L(T_eG^l, TG)$ is $C^{l-1}$, and therefore $(g,p) \mapsto T_eR_g^* p$ from $T^*G$ to $T_e^*G^l$ is also $C^{l-1}$. Since $t \mapsto (g_t, p_t) \in T^*G$ is $C^1$, then the momentum $t \mapsto m_t \in T_e^*G^l$ is $C^{l-1}$.

Let $v \in T_eG^1$, we prove that $t \mapsto (m_t \mid \mathrm{Ad}_{g_t}(v)) = (p_t \mid T_eR_{g_t} \mathrm{Ad}_{g_t}(v))$ is constant. We suppose that $I$ is small enough so that we can make all computations in canonical coordinates on $T^*G$ near $(g_t, p_t)$ (i.e. we locally identify $T^*G$ with the trivial bundle $\mathbb{B} \times \mathbb{B}^*$ where $\varphi : U \to \mathbb{B}$ is a local chart around $g_t$ for $t = t_0$). By definition of the adjoint representation, we get that

$$(m_t \mid \mathrm{Ad}_{g_t}(v)) = (p_t \mid T_eL_{g_t}v)$$

which is $C^1$. Therefore we have

$$\frac{d}{dt}(p_t \mid T_eL_{g_t}v) = \left(p_t \mid \frac{d}{dt}(T_eL_{g_t}v)\right) + (\dot{p}_t \mid T_eL_{g_t}v)$$

We recall that the multiplication

$$\mathrm{Mult}^{2,0} : \left\{ \begin{array}{ccc} G^2 \times G & \longrightarrow & G \\ (g, g') & \longmapsto & gg' \end{array} \right.$$

is $C^2$. We start by computing the derivative $\partial_1\partial_2 \mathrm{Mult}^{2,0}|_{(e,g_t)}\big(u_t, T_eL_{g_t}v\big)$. We have, for $g \in G^2$:

$$\begin{aligned} \partial_2 \mathrm{Mult}^{2,0}|_{(g,g_t)} T_eL_{g_t}(v) &= T_gL_gT_eL_{g_t}v \\ &= T_eL_{gg_t}v \end{aligned}$$

Note that, by $(G.4)$, the map $g \in G^1 \mapsto T_eL_{gg_t}v \in TG$ is $C^1$ and its derivative in the direction $u_t \in T_eG^2$ coincides with the derivative of $g \mapsto T_eL_{gg_t}$ restricted to $G^2$. Since the curve $s \mapsto g_{s+t}g_t^{-1}$ is $C^1$ in $G^1$ with derivative $u_t \in T_eG^2$ at $s = 0$, we get

$$\begin{aligned} \partial_1\partial_2 \mathrm{Mult}^{2,0}(e, g_t)(u_t, T_eL_{g_t}v) &= \partial_g(T_eL_{gg_t}v)|_{g=e}(u_t) \\ &= \frac{d}{ds}(T_eL_{g_{t+s}g_s^{-1}g_t}v)|_{s=0} \\ &= \frac{d}{dt}(T_eL_{g_t}v) \end{aligned}$$

By Schwarz theorem, we also have

$$\begin{aligned} \partial_1\partial_2 \mathrm{Mult}^{2,0}(e, g_t)(u_t, T_eL_{g_t}v) &= \partial_2\partial_1 \mathrm{Mult}^{2,0}(e, g_t)(u_t, T_eL_{g_t}v) \\ &= \partial_g\left(T_eR_g(u_t)\right)|_{g=g_t}T_eL_{g_t}v \end{aligned}$$

Therefore we have

$$\left(p_t \mid \frac{d}{dt}(T_eL_{g_t}v)\right) = (p_t \mid \partial_g\left(T_eR_g(u_t)\right)|_{g=g_t}T_eL_{g_t}v) \tag{5.9}$$

Since $p_t$ is solution of Hamiltonian equation, we get for $\delta g \in T_{g_t}G$ :

$$(\dot{p}_t \mid \delta g) = -\partial_g\mathcal{H}(g_t, p_t, u_t)\delta g = \big(p_t \mid \partial_g(T_eR_g(u_t))_{|g=g_t}\delta g\big)\,.$$

Now for $\delta g = T_eL_{g_t}(v)$ :

$$(\dot{p}_t \mid T_eL_{g_t}v) = -\big(p_t \mid \partial_g(T_eR_g(u_t))_{|g=g_t}(T_eL_{g_t}(v))\big) \tag{5.10}$$

By summing 5.9 and 5.10 we therefore have

$$\forall v \in T_eG^1,\ \frac{d}{dt}\left(m_t \mid \mathrm{Ad}_{g_t}(v)\right) = 0$$

i.e. $m_t = \mathrm{Ad}^*_{g_t^{-1}} m_0$, with $m_0 = p_0$. This equality is understood in $T_e^*G^1$. Indeed, by proposition 4.15, the map $\mathrm{Ad}^{-1} : G^1 \times T_eG^1 \to T_eG$ is $C^1$, and thus induces a continuous morphism (3.21) $\mathrm{Ad}^{-1} : G^1 \to L(T_eG^1, T_eG)$. Consequently the dual mapping $(\mathrm{Ad}^{-1})^* : G^1 \to L(T_e^*G, T_e^*G^1)$ is also continuous. Moreover, since $p_0 \in T_e^*G$, both $t \mapsto m_t \in T_e^*G^1$ and $t \mapsto \mathrm{Ad}^*_{g_t^{-1}} m_0 \in T_e^*G^1$ are continuous and equal. This equality is also true in $T_e^*G^2$ and in such case we can differentiate it (since $m_t \in C^1(I, T_e^*G^2)$) and we get

$$\dot{m}_t + \mathrm{ad}^*_{u_t} m_t = 0.$$

Conversely, suppose $g_t \in \mathrm{AC}_{L^2}(I, G)$ is a horizontal curve, and $t \mapsto m_t \in T_e^*G$ a momentum map satisfying the hypotheses of the theorem. In particular, since $u_t = K^2 m_t$, the curve $t \mapsto u_t$ is $C^1$ and therefore $g_t \in C^1(I, G^1)$. Once again, for all $v \in T_eG^2$,

$$\frac{d}{dt}(m_t \mid \mathrm{Ad}_{g_t}(v)) = 0$$

We define the covector $p_t = (T_eL^*_{g_t})^{-1} p_0 \in C^1(I, T^*G)$ Then $p_t$ is differentiable and we get, for any $v \in T_eG$

$$\begin{aligned}
0 &= \frac{d}{dt}\left(p_t \mid T_eL_{g_t} v\right) \\
&= (\dot{p}_t \mid T_eL_{g_t} v) + (p_t \mid \frac{d}{dt}(T_eL_{g_t} v)) \\
&= (\dot{p}_t \mid \partial_g(T_eR_g(u_t))|_{g=g_t} T_eL_{g_t} v)
\end{aligned}$$

since as we saw earlier $\frac{d}{dt} T_eL_{g_t} v = \partial_g(T_eR_g(u_t))|_{g=g_t} T_eL_{g_t} v$. In particular if we define $\tilde{m}_t = T_eR^*_{g_t} p_t$, we see that once again, for all $v \in T_eG^2$

$$\frac{d}{dt}(\tilde{m}_t \mid \mathrm{Ad}_{g_t}(v)) = 0$$

and in particular $(\tilde{m}_t \mid v) = (m_t \mid v)$, so that $K^2\tilde{m}_t = K^2 m_t = u_t$. Therefore, $t \mapsto (g_t, p_t, u_t)$ solves the Hamiltonian equations. $\square$

Note that in the result we actually proved that the momentum map $m_t = T_eR^*_{g_t} p_t$ satisfies the sub-Riemannian Euler-Arnold-Poincaré equations

**Corollary 5.15** (Integrated version of Euler-Arnold equation). *Suppose $V \hookrightarrow T_eG^2$. Let $t \mapsto (g_t, p_t) \in C^1(I, T^*G)$ be a Hamiltonian flow of* (5.6) *and let $t \mapsto m_t = T_eR_{g_t}^* p_t$ be the associated momentum trajectory in $T_e^*G$. Then:*

1. *the momentum map trajectory verifies (in $T_e^*G^1$)*
$$m_t = \mathrm{Ad}^*_{g_t^{-1}}(m_0)\,; \tag{5.11}$$
2. *the trajectory $t \mapsto g_t$ is uniquely defined as the solution in $G^1$ starting at $e$ of the autonomous differential system*
$$\dot{g} = G_{m_0}(g) \doteq K_V\,\mathrm{Ad}^*_{g^{-1}}(m_0) \cdot g \tag{5.12}$$
*with $G_{m_0} : G^1 \to TG^1$ continuously differentiable.*

*Proof.* The first point is contained in the proof of 5.13. Since $\dot{g}_t = T_eR_{g_t}u_t = T_eR_{g_t}K^1 m_t$, and since $m_t$ is solution of (5.11) the flow $g_t$ is therefore solution of the ordinary equation
$$\dot{g}_t = G_{m_0}(g_t) \tag{5.13}$$
on Banach space $G^1$. We next prove that $G_{m_0} : G^1 \to TG^1$ is $C^1$. We have that for $g \in G^1$,
$$\mathrm{Ad}_{g^{-1}} = T_gL_{g^{-1}} \circ T_eR_g = (T_eL_g)^{-1} \circ T_eR_g$$
By proposition 4.7, the mapping $g \in G^1 \mapsto T_eL_g \in L(T_eG, TG)$ is $C^1$. In a local chart of the bundle $TG \to G^1$ around $g \in G^1$, $g \mapsto T_eL_g$ is the mapping $g \mapsto (g, \widehat{T_eL_g}) \in V \times GL(T_eG)$. As $GL(T_eG)$ is a Banach Lie group, the inverse mapping $\mathrm{inv} : GL(T_eG) \to GL(T_eG)$ is smooth. Thus, the mapping $g \in G^1 \mapsto (T_eL_g)^{-1} \in L(TG, T_eG)$ is also $C^1$. Furthermore, by (G.4), the mapping $(g, u) \in G^1 \times T_eG^2 \mapsto T_eR_gu \in TG$ is $C^2$, and thus by proposition 3.21, $g \in G^1 \mapsto T_eR_g \in L(T_eG^2, TG)$ is $C^1$ and thus locally-Lipschitz. By bilinearity of the composition on the product $L(TG, T_eG) \times L(T_eG^2, TG^k)$, the mapping $g \in G^1 \mapsto \mathrm{Ad}_{g^{-1}} \in L(T_eG^2, T_eG)$ and then the dual mapping $g \in G^1 \mapsto \mathrm{Ad}^*_{g^{-1}} \in L(T_e^*G, T_e^*G^2)$ are also $C^1$. Therefore
$$\begin{array}{rcl} G^1 & \longrightarrow & T_eG^2 \\ g & \longmapsto & K^2\,\mathrm{Ad}^*_{g^{-1}}(m_0) \end{array}$$
is $C^1$. Now, since $T_eR : T_eG^2 \times G^1 \mapsto TG^1$ is $C^1$, then $G_{m_0}$ is $C^1$, and by Picard-Lindelof, the equation (5.12) has a unique solution. □

## 5.3 Completeness theorem

In infinite-dimensional Riemannian geometry, the Hopf-Rinow theorem does not hold anymore [13, 61], even with strong metrics. For right-invariant strong Riemannian metrics defined on Banach half-Lie groups, we can still recover most of the completeness properties [16]. We sum-up results we proved in the previous sections for the strong right-invariant sub-Riemannian metrics on half-Lie groups

**Theorem 5.16** (Completeness properties)**.** *Let $\{G^k, k \geq 0\}$ be an admissible graded group structure 4.3, with $G = G^0$. Let $V \hookrightarrow T_eG$ be a Hilbert space, and $d_V : G \times G \to \mathbb{R} \cup \{\infty\}$ the induced sub-Riemannian distance. Then, the following properties hold:*

1. *The space $(G, d_V)$ is metrically complete.*

2. *Assume that the endpoint mapping* End $: L^2(I, V) \to G$ *is weakly-continuous with regards to some Hausdorff topology on $G$. Then, for any $g_0, g_1 \in G$ such that $d_V(g_0, g_1) < \infty$, there exists a minimizing geodesic connecting $g_0$ and $g_1$.*

3. *Assume $V$ is continuously embedded in $T_eG^2$. Then, for every $p_0 \in T_e^*G$, there exists a unique global curve $(g, p) \in \mathrm{AC}_{L^2}(I, T^*G)$ satisfying the normal geodesic equation with initial covector $p_0$. In other words, the normal Hamiltonian geodesic flow is well-defined and global.*

4. *Assume $V$ is continuously embedded in $T_eG^2$. Then, the Euler-Arnold equation* (5.8) *is globally well-posed, i.e. for any $m_0 \in T_e^*G$, there exists a unique solution in $C(I, T_e^*G^1) \cap C^1(I, T_e^*G^2)$ starting from $m_0$, of the equation*

$$\begin{cases} \dot{m}_t + \mathrm{ad}_{u_t}^* m_t = 0 \\ K^2 m_t = u_t \end{cases}$$

*Proof.* First and second points come from theorem 5.7, and third point is 5.12. Last point comes from propositions 5.13 and 5.15. Indeed, for $p_0 \in T_e^*G$ we can integrate the Hamiltonian flow and get $(g_t, p_t) \in C^1(I, T^*G)$, with $g_t \in C^1(I, G^1)$. Then we saw that the momentum $m_t = T_e R_{g_t}^* p_t$ is a solution (5.8). Conversely, if $m_t \in C(I, T_e^*G^1) \cap C^1(I, T_e^*G^2)$ is solution of (5.8), then $m_t$ is solution of the integrated version, i.e. $\forall v \in T_eG^2$,

$$\frac{d}{dt} (m_t \mid \mathrm{Ad}_{g_t}(v)) = 0$$

where $g_t$ is the unique solution of

$$\dot{g}_t = G_{m_0} g_t, \quad g_0 = e,$$

In particular $m_t = \mathrm{Ad}_{g_t^{-1}}^* m_0$, so we have uniqueness. $\square$

# Chapter 6

# Shape spaces and induced metrics

## Contents



In this chapter, we come back to the LDDMM framework and we extend it using general half-Lie group actions. We introduce a Banach manifold of shapes $\mathcal{Q}$, an admissible graded group structure $\{G^k, k \geq 0\}$, and a Hilbert space $V \hookrightarrow T_eG^2$, inducing a right-invariant sub-Riemannian structure on $G$. We assume that $G$ acts on $\mathcal{Q}$ and we define differentiability asumptions on this action to induce a sub-Riemannian structure on $\mathcal{Q}$. We prove completeness of such structure (metric completeness, geodesic connectedness, global existence of the geodesics).

Moreover, following the LDDMM framework, this sub-Riemannian structure allows us to define an inexact matching problem between a source shape and a target shape, where we add a data attachment term in the variational problem. In particular, we will see that the critical points of this variational problems are normal geodesics for the sub-Riemannian structure, and we will give a Hamiltonian formulation to characterize these critical points. Finally, since the induced metric on the shape spaces $\mathcal{Q}$ is also invariant by the action of the half-Lie group, we retrieve the sub-Riemannian Euler-Poincaré-Arnold equations for the momentum, allowing to lift the geodesics to the half-Lie group. We give some applications in the end of the chapter.

Note that most of the proofs of the results are really similar to the proofs of chapter 5, and we will often refer to [82] for the detailed proofs.

## 6.1 Shape space definition and regularity assumptions

Let $\mathcal{Q}$ be a Banach manifold and $\{G^k, k \geq 0\}$ be an admissible graded group structure 4.3. Let $V$ be a Hilbert space continuously embedded in $T_eG^2$. We suppose $G$ acts on $\mathcal{Q}$ and we denote $A$ the action:

$$A : \begin{cases} G \times \mathcal{Q} & \longrightarrow & \mathcal{Q} \\ (g,\ q) & \longmapsto & g \cdot q \end{cases}$$

We suppose that the following conditions are satisfied :

**(S.1)** *Continuity of the action* : $A : (g, q) \mapsto g \cdot q$ is continuous

**(S.2)** *Infinitesimal action* : For all $q \in \mathcal{Q}$, the mapping $A_q = A(\cdot, q) : g \mapsto g \cdot q$ is $C^\infty$, and we denote $\xi_q = \xi(\cdot, q) = \partial_g A(g, q)|_{g=e}$ its continuous differential in $e$.

**(S.3)** *Regularity of the action* : For $l > 0$, the mappings

$$A : \begin{cases} G^k \times \mathcal{Q} & \longrightarrow & \mathcal{Q} \\ (g,\ q) & \longmapsto & g \cdot q \end{cases} \quad \text{and} \quad \xi : \begin{cases} T_eG^k \times \mathcal{Q} & \longrightarrow & T\mathcal{Q} \\ (u,\ q) & \longmapsto & \xi_q(u) = u \cdot q \end{cases}$$

are $C^k$.

**Remark 6.1.** *Since $V \hookrightarrow T_eG^2$, condition (S.3) implies that the mapping $\xi : V \times \mathcal{Q} \to T\mathcal{Q}$ is also a $C^1$-vector bundle morphism (in the sense of [61], cf. 3.1.2), meaning that $q \in \mathcal{Q} \mapsto \xi_q \in \mathcal{L}(V, T\mathcal{Q})$ is $C^1$. Furthermore its derivative is locally lipschitz. If $G$ is a finite dimension Lie group, then $\xi$ is even a $C^2$-vector bundle morphism, but this property does not hold in infinite dimension.*

In the following, we consider the general framework of a graded group structure acting on a Banach manifold that encompasses both (G.1-5) and (S.1-3) conditions :

**(GGA)** $\{G^k, k \geq 0\}$ is an admissible graded group structure satisfying the conditions (G.1-5), $\mathcal{Q}$ is a Banach manifold playing the role of a *shape space* and $G$ acts on $\mathcal{Q}$ and satisfies conditions (S.1-3) with $V$ a Hilbert space continuously embedded in $T_eG^2$.

**Remark 6.2.** *In particular, the multiplication of the group $G$ induces a left action that verifies $(S.1-3)$ conditions. Therefore, the family of groups $\{G^k, k \in \mathbb{N}\}$ endowed with its natural left action verifies the (GGA) conditions, with $\mathcal{Q} = G$ and $V \hookrightarrow T_eG^2$. In this case, the infinitesimal action is simply the derivative of the right multiplication :*

$$\xi_g(u) = T_eR_g(u) = u \cdot g$$

**Example 6.3** (LDDMM framework). *In particular, the setting defined in [6] particularly fits in the conditions described here. It gives the geometry of the LDDMM framework, that was first defined in [95]. Here the authors consider an action of the half-Lie group* $\mathrm{Diff}_{H^s}(\mathbb{R}^d)$ *(with $s \geq d/2 + 1$) on a Banach manifold $\mathcal{Q}$ satisfying conditions (S.1-3). Here are some of the main examples :*

- *Space of landmarks : Let $n > 0$. We consider the space of landmarks*

$$\mathrm{Lmk}_n(\mathbb{R}^d) = \{(q_i)_{i \leq n} \in (\mathbb{R}^d)^n \mid q_i \neq q_j \text{ if } i \neq j\},$$

*that is to say the space of collections of n-points of* $\mathbb{R}^d$. *Then the group of diffeomorphisms acts naturally on* $\mathrm{Lmk}_n$ *by transporting each point :*

$$\varphi \cdot (q_i) = (\varphi(q_i))$$

*with infinitesimal action given by* $v \cdot (q_i)_i = (v(q_i))_i$.

- *Space of curves : Let D denotes either the interval* $[0,1]$, *for open curves, or the circle* $\mathbb{S}^1$ *for closed curves. We define the space of curves as the space of* $H^1$*-immersions*

$$\mathrm{Imm}_{H^1}(D, \mathbb{R}^d) = \{q \in H^1(D, \mathbb{R}^d) \mid q'(t) \neq 0 \textit{ for all } t \in D\}.$$

*It is an open subset of the Hilbert space* $H^1(D, \mathbb{R}^d)$ *which induces a manifold structure with tangent space* $T_q \mathrm{Imm}_{H^1}(D, \mathbb{R}^d) \simeq H^1(D, \mathbb{R}^d)$, *the space of* $H^1$ *vector fields along* $q$. *The group of diffeomorphisms acts on the curves by left composition :*

$$(\varphi \cdot q)(t) = \varphi \circ q(t), \quad t \in D.$$

**Example 6.4** (The special case of images). *In computational anatomy and medical imaging, we are particularly interested in the matching of a template image* $I_S \in \mathcal{I}$ *onto a target image* $I_T \in \mathcal{I}$ *by a diffeomorphism. We consider the space* $\mathcal{I} := L^2(\Omega, \mathbb{R})$ *of grey scale images on the image domain* $\Omega \subset \mathbb{R}^d$, *and the group of diffeomorphisms* $\mathrm{Diff}_{H^s}(\Omega)$ *acts by right composition by the inverse:*

$$\varphi \cdot I = I \circ \varphi^{-1}.$$

*Because of the right composition, there is again a loss of derivative and when the image I is fixed, the map* $\varphi \mapsto \varphi \cdot I$ *is not smooth. In particular, the action on the space of images does not satisfy the conditions (S.1-3). We can still retrieve some of the results of this chapter by playing with the regularity of the source image* $I_S$. *Similarly the action of the group of diffeomorphisms on the space of varifolds as defined in [21] does not satisfy either the (S.1-3) regularity properties : let* $p \geq 0$, *and consider* $\mathcal{G}_p(\mathbb{R}^d)$ *the* $p-$*Grassmannian space of* $\mathbb{R}^d$

$$\mathcal{G}_p(\mathbb{R}^d) = \{V \subset \mathbb{R}^d \textit{ vector subspace} \mid \dim(V) = p\}.$$

*The space of* $p-$*varifolds is then the space of Radon measures on* $\mathbb{R}^d \times \mathcal{G}_p(\mathbb{R}^d)$ *:*

$$\mathcal{M}_s(\mathbb{R}^d \times \mathcal{G}_p(\mathbb{R}^d)) = \mathcal{C}_0(\mathbb{R}^d \times \mathcal{G}_p(\mathbb{R}^d))',$$

*and the group of diffeomorphisms acts on the space of p-varifolds by*

$$(\varphi \cdot \mu \mid \omega) = \int_{\mathbb{R}^d \times \mathcal{G}_p(\mathbb{R}^d)} \omega\left(x, \omega(\varphi(x), d\varphi(x)V\right) |J_{\varphi|V}(x)| d\mu(x, V), \quad \forall \omega \in \mathcal{C}_0(\mathbb{R}^d \times \mathcal{G}_p(\mathbb{R}^d)).$$

*Then this action satisfies the same loss of derivative property as the action of diffeomorphisms on images.*

## 6.2 Induced sub-Riemannian geometry

### 6.2.1 Induced right-invariant structure

Let $\mathcal{Q}$ be a Banach manifold, $\{G^k,\, k \geq 0\}$ an admissible graded group structure, with $G$ acting on $\mathcal{Q}$ following the (GGA) framework 6, with $V \hookrightarrow T_e G^2$. The triple $(\mathcal{Q} \times V, \xi, \langle \cdot, \cdot \rangle_V)$ now defines a sub-Riemannian structure on $\mathcal{Q}$:

- $\mathcal{Q} \times V$ is a vector bundle over $\mathcal{Q}$.
- The map
$$\xi : \begin{array}{ccc} \mathcal{Q} \times V & \longrightarrow & T\mathcal{Q} \\ (q, u) & \longmapsto & \xi_q u \end{array}$$
is a $C^1$-vector bundle morphism
- The metric $\langle \cdot, \cdot \rangle_V$ is the scalar product from the Hilbert space $V$, which induces a metric on the bundle $\mathcal{Q} \times V$ :
$$\langle u, v \rangle_q = \langle u, v \rangle_V, \quad q \in \mathcal{Q},\ u, v \in V$$

This defines a horizontal distribution $\Delta \subset T\mathcal{Q}$ and each horizontal space $\Delta_q = \xi_q(V)$ can be endowed with a Hilbert structure by defining a scalar product $\langle \cdot, \cdot \rangle_q$:

$$\langle X, X \rangle_q = \inf_{u \in V,\ \xi_q u = X} \langle u, u \rangle_V = \langle \xi_q^{-1} X, \xi_q^{-1} X \rangle_V$$

where $\xi_q^{-1}$ is the inverse of the restriction $\xi_q : (\ker \xi_q)^\perp \to \Delta_q$, and $(\ker \xi_q)^\perp$ is the orthogonal of $\ker \xi_q$ in $V$. We get the following

**Proposition 6.5** (Reduction principle)**.** *Let $q \in \mathcal{Q}$, and suppose $\Delta_q$ is closed in $T_q\mathcal{Q}$. Let $K_V : V^* \to V$ be the inverse of the Riesz canonical isometry. The orthogonal $(\ker \xi_q)^\perp$ of $\ker \xi_q$ in $V$ is simply given by*

$$(\ker \xi_q)^\perp = \{K_V \xi_q^* p \mid p \in T_q^*\mathcal{Q}\}$$

**Remark 6.6.** *This also justifies in particular the introduction of the co-metric in 6.3 to study the normal sub-Riemannian geodesics in $\mathcal{Q}$. Note also that the condition of $\Delta_q$ being closed in $T_q\mathcal{Q}$ can also be quite restrictive when the manifold $\mathcal{Q}$ is infinite dimensional.*

*Proof.* If $u = K_V \xi_q^* p$ with $p \in T_q^*\mathcal{Q}$, we immediately have that for any $v \in \ker \xi_q$,

$$\langle u, v \rangle_V = (\xi_q^* p \mid v) = (p \mid \xi_q v) = 0$$

and $u \in (\ker \xi_q)^\perp$. Conversely suppose $u = K_V m$ is in $(\ker \xi_q)^\perp$, with $m \in V^*$. We define for any $X \in T_q\mathcal{Q}$,
$$(p \mid X) = \langle u, \xi_q^{-1} X \rangle_V,$$
that is to say we have $p = \xi_q^{-1*} m \in T_q^*\mathcal{Q}$ (the map $p$ is indeed continuous since the restriction $\xi_q : (\ker \xi_q)^\perp \to \Delta_q$ is bijective continuous onto the Banach closed subspace $\Delta_q$, and by using the Banach inverse theorem, the inverse $\xi_q^{-1}$ is also continuous). Moreover, for any $v \in V$, we have

$$\begin{aligned} \langle K_V \xi_q^* p, v \rangle_V &= (\xi_q^* p \mid v) \\ &= (p \mid \xi_q v) \\ &= \langle u, \xi_q^{-1} \xi_q v \rangle_V \quad \text{by definition of } p \\ &= \langle u, v \rangle_V \quad \text{since } u \in (\ker \xi_q)^\perp \end{aligned}$$

and thus $u = K_V \xi_q^* p$, which concludes the proof. □

### 6.2.2 Horizontal curves and sub-Riemannian distance

In this part, we denote by $I$ a closed interval of $\mathbb{R}$, and we suppose the hypotheses of the (GGA) framework are satisfied.

**Definition 6.7** (Horizontal curves in $\mathcal{Q}$)**.** An absolutely continuous curve $q : I \to \mathcal{Q}$ is said to be *horizontal* if there exists a continuous lift $t \mapsto u(t) \in V$ such that

$$\forall t \in I,\ \dot{q}(t) = \xi_{q(t)}(u(t))$$

We call an *horizontal system* such a couple $(q, u) \in AC_{L^1}(I, \mathcal{Q}) \times L^1(I, V)$.

**Remark 6.8.** *If $q : I \to \mathcal{Q}$ is a horizontal curve in $\mathcal{Q}$, there can exist two different controls $u_1 \neq u_2$ such that*

$$\dot{q} = \xi_q(u_1) = \xi_q(u_2)$$

For such horizontal curves $q : I \to \mathcal{Q}$, we can define the corresponding *minimal lift* $g \in AC(I, G)$ as the only curve satisfying

$$\dot{g}_t = T_e R_{g_t}\big(\xi_{q_t}^{-1}\dot{q}_t\big), \quad g_0 = e$$

Note that there can exist other horizontal lifts in $G$ of the same curve in $\mathcal{Q}$, and the minimal lift is the lift that minimizes the energy in $G$ :

$$E(g_t, \xi_{q_t}^{-1}\dot{q}_t) = \min_{u_t \in L^2(I,V),\, \mathrm{Evol}(u_t)\cdot q_0 = q_t} E(\mathrm{Evol}(u_t), u_t) \tag{6.1}$$

Similarly to propositions 4.18, we can define an evolution map in $\mathcal{Q}$ by solving the corresponding ordinary differential equation.

**Theorem 6.9** (Evolution map in $\mathcal{Q}$ and regularity)**.** *Let $q_0 \in \mathcal{Q}$, and $u \in L^2(I, V)$. There exists a unique $q \in AC_{L^2}(I, \mathcal{Q})$ such that $q(0) = q_0$ and*

$$\dot{q}(t) = \xi_{q(t)}\big(u(t)\big) \ a.e. \tag{6.2}$$

*Furthermore the evolution map in $\mathcal{Q}$*

$$\mathrm{Evol}_{\mathcal{Q}} : u \in L^2(I, V) \mapsto q^u \in AC_{L^2}(I, \mathcal{Q})$$

*where $q^u$ is solution of equation* (6.2)*, is $C^1$. For $u, \delta u \in L^2(I, V)$, its derivative $\delta q = T_u \mathrm{Evol}_{\mathcal{Q}} \cdot \delta u$ is solution of the linear Cauchy problem*

$$\delta\dot{q}(t) = \partial_q\left(\xi_q(u(t))\right)_{|q=q^u(t)} \delta q(t) + \partial_u(\xi_{q^u(t)}(u))_{|u=u(t)}\delta u(t),\ \delta q(0) = 0 \tag{6.3}$$

*Proof.* This follows from Proposition 4.18. As $u \in L^2(I, V)$, and $\xi : V \times \mathcal{Q} \to T\mathcal{Q}$ is $C^2$ (S.3), there exists by Picard-Lindelof one unique maximal (not necessarily global) solution $q$ of equation (6.2). There also exists by proposition 4.18 a unique $g = \mathrm{Evol}_G(u) \in AC_{L^2}(I, G)$ such that

$$\dot{g}(t) = u(t) \cdot g(t), \quad g(0) = e$$

It is immediate to verify that $t \mapsto A(g(t), q_0) = g(t) \cdot q_0$ is also solution to equation (6.2), and thus $q(t) = A(g(t), q_0)$. Therefore we have $\mathrm{Evol}_{\mathcal{Q}} = \tilde{A}_{q_0} \circ \mathrm{Evol}_G$, where $\tilde{A}_{q_0}$ is given by :

$$\tilde{A}_{q_0} : \begin{cases} AC_{L^2}(I,G) & \longrightarrow & AC_{L^2}(I,\mathcal{Q}) \\ g & \longmapsto & [t \mapsto A_{q_0}(g(t))] \end{cases}$$

By (S.2), the mapping $A_{q_0}$ is smooth, and therefore the induced mapping $\tilde{A}_{q_0}$ is also smooth [41, lemma 3.27], with derivative $T_g\tilde{A}_{q_0} : AC_{L^2}(I,TG) \to AC_{L^2}(I,T\mathcal{Q})$, $g \in AC_{L^2}(I,G)$, given by

$$T_g\tilde{A}_{q_0} : \delta g \mapsto [t \mapsto T_{g(t)}A_{q_0}(\delta g(t)) = \xi_{g(t)\cdot q_0} \circ (T_eR_{g(t)})^{-1}(\delta g(t))]$$

Now as $L^2(I,V) \hookrightarrow L^2(I,T_eG^2)$ smoothly, then the evolution mapping in $\mathcal{Q}$ is also $C^1$ by composition. For $u, \delta u \in L^2(I,V)$, and $\delta q = T_u \operatorname{Evol}_{\mathcal{Q}}(\delta u) = T_{\operatorname{Evol}_G(u)}\tilde{A}_{q_0} \circ T_u \operatorname{Evol}_G(\delta u)$ we thus have

$$\delta q(t) = \xi_{q^u(t)} \circ (T_eR_{g(t)})^{-1}(\delta g(t))$$

where $g = \operatorname{Evol}_G(u)$, and $\delta g = T_u \operatorname{Evol}_G(\delta u)$ satisfies equation (4.8). By derivation of this equation in charts (or adapting the proof of proposition 4.22 and introducing the mapping $q, u \mapsto \dot{q} - \xi_q u$) we get that $\delta q$ is solution of (6.3). $\square$

In the following, we will denote

$$\operatorname{Hor}_{q_0}(I) = \{(q,u) \in AC_{L^2}(I,\mathcal{Q}) \times L^2(I,V),\ q = \operatorname{Evol}_{\mathcal{Q}}(u) \text{ and } q(0) = q_0\}$$

the space of all horizontal systems with starting point $q_0$. It is in general not a submanifold of $AC_{L^2}(I,\mathcal{Q}) \times L^2(I,V)$ but it's in bijection with $L^2(I,V)$. Now we define the energy and length of an horizontal system : Let $(q,u) : I \to \mathcal{Q} \times V$ a horizontal system. We define its *length* and *energy* respectively by

$$L(q,u) = \int_I |u(t)|_V dt \quad \text{and} \quad E(q,u) = \tfrac{1}{2}\int_I |u(t)|_V^2 dt$$

We can therefore define the sub-Riemannian distance induced by $V$ on $\mathcal{Q}$ as the infimum of the length of horizontal curves :

**Definition 6.10** (Sub-Riemannian distance)**.** Let $q_0, q_1 \in \mathcal{Q}$. We define the sub-Riemannian distance $d_V(q_0,q_1)$ as

$$d_V(q_0,q_1) = \inf_{\substack{(q,u) \text{ horizontal} \\ q(0)=q_0, q(1)=q_1}} L(q,u)$$

Once again, we obtain true distance on $\mathcal{Q}$

**Proposition 6.11.** *The application $d_V$ is a true distance on $\mathcal{Q}$, and the topology induced by the sub-Riemannian distance is weaker than the intrinsic topology on $\mathcal{Q}$. Furthermore the distance $d_V$ is also equal to the infimum of the energy on horizontal curves, i.e. for $q_0, q_1 \in \mathcal{Q}$ :*

$$d_V(q_0,q_1) = \inf_{\substack{(q,u) \text{ horizontal} \\ q(0)=q_0, q(1)=q_1}} \sqrt{2E(q,u)}$$

*where we take the infimum over the set $AC_{L^2}(I,\mathcal{Q}) \times L^2(I,V)$.*

*Proof.* The proof is mostly contained in [7], and similar to the proof of 5.16. We refer to the appendix [82, proposition 3.8] for the detailed proof. □

We finish the section with some geodesic and metric completeness result. We first recall the definition of geodesics associated with the distance $d_V$ :

**Definition 6.12** (Geodesics in $\mathcal{Q}$)**.** Let $(q, u) \in AC_{L^2}(I, \mathcal{Q}) \times L^2(I, V)$ an horizontal system. Then

- We say that the curve $(q, u)$ is a *geodesic* if it minimizes locally the length, meaning for every $t_0 \in I$, and $t_1$ close enough to $t_0$ :

$$L((q, u)_{|[t_0,t_1]}) = d_V(q(t_0), q(t_1))\,.$$

- The curve $(q, u)$ is a *minimizing geodesic* if its total length is equal to the distance between the endpoints.

**Remark 6.13.** *We already saw in proof of Proposition 6.11 that if $(q, u)$ is a horizontal system that minimizes the energy, then it's immediately a minimizing geodesic, and is also parametrized with constant speed. Conversely, if $(q, u)$ is a minimizing geodesic parametrized with constant speed, then $(q, u)$ also minimizes the energy, and we have*

$$L(q, u) = \sqrt{2E(q, u)}\,.$$

We will need another assumption on the action of the groups $\{G^k, k\}$ to prove that $(\mathcal{Q}, d_V)$ is a geodesic metric space, i.e. that we can join any two points of $\mathcal{Q}$ by a minimizing geodesic :

**(S.4)** For every $q \in \mathcal{Q}$, the endpoint mapping $\mathrm{End}_q : L^2(I, V) \to \mathcal{Q}, u \mapsto A_q \circ \mathrm{Evol}_G(u)(1)$ is weakly continuous where $\mathcal{Q}$ is equipped with some Hausdorff topology

**Remark 6.14.** *In most cases, the endpoint mapping* $\mathrm{End} : L^2(I, V) \to G$ *in $G$ is weakly continuous with regards to some Hausdorff topology in $G$, and thus we just need to study continuity of mapping $A_q$.*

**Theorem 6.15** (Metric completeness)**.** *1. The space $\mathcal{Q}$ with distance $d_V$ is metrically complete.*

*2. Moreover, if (S.4) is satisfied, then $\mathcal{Q}$ is a geodesic metric space, meaning for $q, q' \in \mathcal{Q}$ such that $d_V(q, q') < \infty$, there exists a minimizing geodesic connecting $q$ and $q'$.*

*Proof.* The proof is similar to the proof of theorem 5.7. We refer to [82, theorem 3.12] for the detailed proof. □

As in section 5.1.3, we can define three types of sub-Riemannian geodesics associated with $V$. We focus only on the normal case, and we recall the definition of normal geodesic.

**Definition 6.16** (Sub-Riemannian normal geodesic). Let $(q, u) \in \mathrm{Hor}_{q_0}(I)$ an horizontal system. We say that $(q, u)$ is a *sub-Riemannian normal geodesic* (or just normal geodesic) if there exists Lagrange multipliers $(\lambda_0, \lambda) \in \mathbb{R} \times T_{q(1)}\mathcal{Q}^*$, with $\lambda_0$ not equal to zero such that

$$\lambda_0 dE(u)\delta u + (\lambda \mid d\,\mathrm{End}_{q_0}(u)\delta u) = 0 \tag{6.4}$$

## 6.3 Normal geodesics and inexact matching problem

#### 6.3.0.1 A variational problem

Given a source object $q_S$ and a target $q_T$, we want to find the best deformation $g \in G$ matching $q_S$ and $q_T$, and we consider the following inexact problem

$$\inf_{g \in G} d_V(e, g) + \mathcal{D}(g \cdot q_S) \tag{6.5}$$

with $\mathcal{D} : \mathcal{Q} \to \mathbb{R}^+$ a data attachment term that measures the discrepancy between the deformed object $g \cdot q_S$ and the target object $q_T$. Using the definition of the distance $d_V$, we can reformulate the matching problem (6.5) and optimize the functional $J : L^2(I, V) \to \mathbb{R}^+$ defined as

$$J(u) = E(u) + \mathcal{D}(\mathrm{End}_{q_S}(u)). \tag{6.6}$$

If we suppose (S.4) is satisfied, i.e. if in particular the endpoint mapping $\mathrm{End}_{q_S} : L^2(I, V) \to \mathcal{Q}$ is weakly continuous, we can also prove existence of minimizers

**Proposition 6.17** (Existence of minimizers for the inexact matching problem). *Suppose (S.4) is satisfied and $g$ is continuous for the same Hausdorff topology. Then there exists $(q, u) \in \mathrm{Hor}_{q_0}(I)$ such that $J(u)$ is minimal.*

*Proof.* The proof follows the proof of theorem 5.7. We introduce a minimizing sequence $(q_n, u_n) \in \mathrm{Hor}_{q_0}(I)$. The sequence $(u_n)$ converges weakly to $u_\infty \in L^2(I, v)$, and we denote $q_\infty \in AC_{L^2}(I, \mathcal{Q})$ such that $\dot{q}_\infty = \xi_{q_\infty} u_\infty$. Since $\mathrm{End}_{q_0}$ is weakly continuous, and $\mathcal{D}$ is continuous, we get that $\mathcal{D}(q_\infty(1)) = \lim \mathcal{D}(q_n(1))$. Moreover, the lower semi-continuity of the $L^2$ norm gives

$$J(u_\infty) = |u_\infty|_{L^2} + \mathcal{D}(q_\infty(1)) \leq \liminf |u_n|_{L^2} + \lim \mathcal{D}(q_n(1)) \leq \lim \left(|u_n|_{L^2} + \mathcal{D}(q_n(1))\right)$$

Hence the result. □

We can also obtain a characterization of the critical points of $J$ on $L^2(I, V)$

**Theorem 6.18** (Critical points of $J$). *Assume that $\mathcal{D}$ is $C^1$. Let $(q, u) \in \mathrm{Hor}_{q_0}(I)$. Then if $(q, u)$ is a critical point of $J$, it is a normal geodesic.*

*Proof.* Let $(q, u) \in \mathrm{Hor}_{q_0}(I)$ a critical point of $J$, i.e, we have :

$$dJ(u) = dE(u) + d\mathcal{D}(q(1))d\,\mathrm{End}_{q_0}(u) = 0$$

Therefore $(q, u)$ is a normal geodesic with covector $p(1) = d\mathcal{D}\,(q(1))$ □

#### 6.3.0.2 Co-metric and Hamiltonian equations

We recall that $V \hookrightarrow T_eG^2$. As in 5.2, the sub-Riemannian structure defines a cometric

$$K^{\mathcal{Q}} : \begin{cases} T^*\mathcal{Q} & \longrightarrow & T\mathcal{Q} \\ (q,p) & \mapsto & K_q^{\mathcal{Q}} p = \xi_q K_V \xi_q^* \end{cases}$$

and a normal Hamiltonian $H_{\mathcal{Q}}(q,p) = (p \mid K_q^{\mathcal{Q}} p)$. The Hamiltonian $H$ is $C^1$ (as in 5.2.2), and defines a symplectic gradient

$$\nabla^\omega H_{\mathcal{Q}}(q,p) = \left(K_q^{\mathcal{Q}} p, -\partial_q (K_q^{\mathcal{Q}} p)^* p\right).$$

We retrieve the previous characterization of the normal geodesics

**Proposition 6.19** (Normal Hamiltonian equations)**.** *Let $q \in AC_{L^2}(I, \mathcal{Q})$ an horizontal curve. Then $q_t$ is a normal geodesic if and only if there exists $t \mapsto p(t) \in T^*_{q(t)}\mathcal{Q}$ in $AC_{L^2}(I, T^*\mathcal{Q})$ such that $(q,p)$ satisfies the Hamiltonian equations :*

$$(\dot{q}, \dot{p}) = \nabla^\omega H_{\mathcal{Q}}(q,p). \tag{6.7}$$

*Proof.* The proof is similar to the proof of theorem 5.10. We refer to [82, theorem 3.15] for a complete proof. □

We also get existence and uniqueness of the solutions to the Hamiltonian equations.

**Proposition 6.20** (Existence and uniqueness of the Hamiltonian flow in $\mathcal{Q}$)**.** *We have global existence and uniqueness of the geodesic equation :*

$$(\dot{q}_t, \dot{p}_t) = \nabla^\omega H_{\mathcal{Q}}(q_t, p_t)$$

*with initial value $(q_0, p_0) \in T^*\mathcal{Q}$. This Hamiltonian flow $t \mapsto (q_t, p_t)$ is $C^1$ on $T^*\mathcal{Q}$.*

*Proof.* We refer to [82, proposition 5.1]. □

## 6.4 Euler Poincaré theory

Reduction theory can be applied again and we introduce a momentum map

$$m_{\mathcal{Q}}(q,p) = \xi_q^* p \in T_e^* G.$$

Since the metric on $\mathcal{Q}$ comes from the right-invariant metric on $G$, normal geodesics on $\mathcal{Q}$ can be lifted as normal geodesics on $G$:

**Theorem 6.21** (Normal geodesics in $\mathcal{Q}$)**.** *Let $q \in AC_{L^2}(I, T\mathcal{Q})$ be a horizontal curve in $\mathcal{Q}$, and let $g \in AC_{L^2}(I, TG)$ be its minimal lift. Then, $q_t$ is a normal geodesic if and only if $g_t$ is a normal geodesic in $G$.*
*Moreover, if $t \mapsto (g_t, p_t^G)$ and $t \mapsto (q_t, p_t^{\mathcal{Q}})$ satisfy the normal Hamiltonian equations, with starting condition*

$$m_G(g_0, p_0^G) = m_{\mathcal{Q}}(q_0, p_0^{\mathcal{Q}}) \tag{6.8}$$

*then $g_t$ is the minimal lift of $q$ and the momentum maps are equal:*

$$\forall t \in I,\ m_{\mathcal{Q}}(q_t, p_t^{\mathcal{Q}}) = m_G(g_t, p_t^G).$$

**Remark 6.22.** *We recall that, by proposition 5.12, the minimal lift $g_t$ thus belongs to $G^1$.*

*Proof.* We refer to [82, Theorem 5.2] for the detailed proof. It uses in particular theorems 5.13 and 5.15, since we can prove that the momentum $m_{\mathcal{Q}}(q,p)$ satisfies the sub-Riemannian Euler-Poincaré-Arnold equations, which characterizes the normal sub-Riemannian geodesics in $G$. □

This tells us that studying the sub-Riemannian normal geodesics in $\mathcal{Q}$ is in a way equivalent to studying the sub-Riemannian normal geodesics in $G$ through the use of the momentum $m_t$. It is particularly interesting since the critical points for the inexact matching problem and the normal geodesics in $\mathcal{Q}$ thus do not depend on the shape space $\mathcal{Q}$, and therefore several parametrizations of shapes lead to the same dynamic :

**Corollary 6.23** (Uniqueness of the momentum map trajectory)**.** *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be Banach manifolds such that $G$ acts on $\mathcal{Q}_1$ and $\mathcal{Q}_2$ within the (GGA) framework. Let $(q_0^1,p_0^1) \in T\mathcal{Q}_1^*$ and $(q_0^2,p_0^2) \in T\mathcal{Q}_2^*$ such that $m_{\mathcal{Q}_1}(q_0^1,p_0^1) = m_{\mathcal{Q}_2}(q_0^2,p_0^2)$. Then for all $t \geq 0$, we have*

$$m_t \doteq m_{\mathcal{Q}_1}(q_t^1,p_t^1) = m_{\mathcal{Q}_2}(q_t^2,p_t^2)$$

*and there is a unique lifted trajectory $t \mapsto g_t$ satisfying*

$$\dot{g}_t = K_V m_t \cdot g_t \text{ and } \begin{cases} q_t^1 = g_t \cdot q_0^1 \\ \\ q_t^2 = g_t \cdot q_0^2 \end{cases}$$

*Proof.* This is a direct consequence of 6.21. Indeed, let $((q_t^1,p_t^1),u_t^1) \in T\mathcal{Q}_1^* \times V$ and $((q_t^2,p_t^2),u_t^2) \in T\mathcal{Q}_2^* \times V$ be normal geodesics satisfying the initial conditions of the corollary, and let $g_t^1$ and $g_t^2$ be their lifts in $G^1$. Then both $g_t^1$ and $g_t^2$ satisfies the differential equation in $G^1$

$$\dot{g}_t = G_{m_0}(g_t)$$

with $g_0^1 = g_0^2 = e_G$, where we recall the map $G_{m_0} : G^1 \to TG^1$ is defined in 5.15 by $G_{m_0}(g) = K_V \operatorname{Ad}_{g^{-1}(m_0)} \cdot g$. This equation admits a unique solution, and therefore $g_t^1 = g_t^2$ and the corollary is proved. □

## 6.5 Extended diffeomorphisms groups and regularity of the covector

The usual path on riemannian shape spaces defined from the induced metric by the action of diffeomorphisms has been following the *reduction* route: one computes the hamiltonian on the shape space $\mathcal{Q}$, derives there shooting algorithms encoded from a given initial momenta in the cotangent space and when needed the lifted trajectory. This happens to be quite effective when the space space $\mathcal{Q}$ supports some sort of discretisation into a finite dimensional space. A common setting is the case of landmarks where spaces are described as point clouds or various triangulations. If the reduction route can be beneficial from a memory point of view since providing finite dimensional encoding of the lifted geodesic in the group of diffeomorphisms, interestingly over-parametrization, or the *unreduction* route, can be also helpful since it may allow a gain of regularity in the representation of the momentum compare to the reduced case. We consider in the following section an illustration of this phenomenon for the group of diffeomorphisms arising in a common situation of inexact matching for transport of atlases.

### 6.5.1 Some extensions of groups of diffeomorphisms

In this part, we define extensions of groups of diffeomorphisms that satisfy the conditions (G.1-5). We first recall the definition of the diffeomorphisms group $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ for $k \geq 1$:

$$G^k = \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) = \left(\mathrm{id} + C_0^k(\mathbb{R}^d, \mathbb{R}^d)\right) \cap \mathrm{Diff}^1(\mathbb{R}^d)\,.$$

Now we introduce two semi-direct products, $H^k$ and $N^k$:

$$H^k = \mathrm{Diff}_{C_0^k} \ltimes C_0^{k-1}(\mathbb{R}^d, \mathbb{R}_{>0})$$

with group operation given for all $(\varphi, \omega), (\varphi', \omega') \in H^k$ by

$$(\varphi, \omega) \cdot (\varphi', \omega') = (\varphi \circ \varphi', \omega \circ \varphi' \omega')$$

and

$$N^k = \mathrm{Diff}_{C_0^k} \ltimes C_0^{k-1}(\mathbb{R}^d, GL_d)$$

with group operation given for all $(\varphi, A), (\varphi', A') \in N^k$ by

$$(\varphi, A) \cdot (\varphi', A') = (\varphi \circ \varphi', A \circ \varphi' A')$$

where $GL_d$ is the general linear group on $\mathbb{R}$.

**Remark 6.24.** *Clearly, $H^k$ and $N^k$ are (normal) extensions of the group $G^k$ so that $G^k$ can be seen as a subgroup of bigger groups (providing a natural setting for over-parametrization).*

In the following, for $\varphi \in G^k$, $J\varphi(x) \in GL_d$ is its Jacobian matrix at $x \in \mathbb{R}^q$ and $|J\varphi|(x)$ its Jacobian determinant.

**Proposition 6.25.** *We have the following properties :*

1. *The families of groups* $(H^k)_{k\geq 1}$, $(N^k)_{k\geq 1}$ *satisfy the conditions (G.1-5).*
2. *The group* $G^k$ *is embedded in both* $H^k$ *and* $N^k$ *through the* ***smooth*** *mappings*

$$i_H : \left| \begin{array}{lll} G^k & \longrightarrow & H^k \\ \varphi & \longmapsto & (\varphi, |J\varphi|) \end{array} \right. \quad \text{and} \quad i_N : \left| \begin{array}{lll} G^k & \longrightarrow & N^k \\ \varphi & \longmapsto & (\varphi, J\varphi) \end{array} \right.$$

*and the differentials of this mappings are given by :*

$$Ti_H : \left| \begin{array}{lll} TG^k & \longrightarrow & TH^k \\ (\varphi, \delta\varphi) & \longmapsto & ((\varphi, |J\varphi|), (\delta\varphi,\ \mathrm{div}\,(\delta\varphi \circ \varphi^{-1}) \circ \varphi\, |J\varphi|)) \end{array} \right.$$

$$Ti_N : \left| \begin{array}{lll} TG^k & \longrightarrow & TN^k \\ (\varphi, \delta\varphi) & \longmapsto & ((\varphi, |J\varphi|), (\delta\varphi, J\delta\varphi)) \end{array} \right. .$$

3. *The mapping* $i_H$ *is a morphism of groups* ($i_H(\varphi' \circ \varphi) = i_H(\varphi') \cdot i_H(\varphi)$) *inducing an action of* $G^k$ *on* $H^k$ *by left multiplication* ($\varphi \cdot h \doteq i_H(\varphi) \cdot h$) *that satisfy (S.1-3) and such that we have for any* $u \in T_eG^k$

$$T_\varphi i_H(u.\varphi) = T_{\mathrm{id}} i_H(u) \cdot i_H(\varphi)\,.$$

*The same is true for* $i_N$ *and* $N^k$.

*Proof.* We already saw $\mathrm{Diff}_{C_0^k}$ satisfies the conditions (G.1-5). Furthermore, $C_0^{k-1}(\mathbb{R}^d, \mathbb{R}_{>0})$ and $C_0^{k-1}(\mathbb{R}^d, GL_d)$ are Lie groups for the pointwise multiplication. Furthermore, for $p, l \in \mathbb{N}$, the mapping

$$\begin{array}{lll} C_0^{k-1+l}(\mathbb{R}^d, \mathbb{R}^p) \times G^k & \longrightarrow & C_0^{k-1}(\mathbb{R}^d, \mathbb{R}^p) \\ (a, \varphi) & \longmapsto & a \circ \varphi \end{array}$$

is $C^l$ and $C^\infty$ with regards to the first variable. For $(\varphi, \omega) \in H^k$ (resp. $(\varphi, A) \in N^k$), its inverse is given by $(\varphi, \omega)^{-1} = (\varphi^{-1}, \frac{1}{\omega \circ \varphi^{-1}})$ (resp. $(\varphi, A)^{-1} = (\varphi^{-1}, A^{-1} \circ \varphi^{-1})$ and (G.2) is verified. Therefore $H^k$ and $N^k$ also satisfy (G.1-5).

We directly see $i_H$ and $i_N$ are smooth morphisms of groups, and are embeddings. Let's compute their derivatives. For $(\varphi, \delta\varphi) \in TG^k$ we get

$$\begin{aligned} \partial_\varphi (|J\varphi|)\, \delta\varphi &= |\det(d\varphi)|\, \mathrm{tr}(d\varphi^{-1} d\delta\varphi) \\ &= |J\varphi|\, \mathrm{tr}\left(d\left(\delta\varphi \circ \varphi^{-1}\right) \circ \varphi\right) \\ &= |J\varphi|\, \mathrm{div}(\delta\varphi \circ \varphi^{-1}) \circ \varphi \end{aligned}$$

Thus $T_\varphi i_H(\delta\varphi) = (\delta\varphi,\ \mathrm{div}\,(\delta\varphi \circ \varphi^{-1}) \circ \varphi\, |J\varphi|)$ and $T_\varphi i_N(\delta\varphi) = (\delta\varphi, J\delta\varphi)$.

The last point follows from the chain rule $|J(\varphi' \circ \varphi)| = |J\varphi'| \circ \varphi |J\varphi|$ which implies $i_H(\varphi' \circ \varphi) = i_H(\varphi') \cdot i_H(\varphi)$ ☐

Since the family of groups $G^k$ act on $H^k$ and $N^k$ satisfying (S.1-3), we can apply proposition 6.21 and corollary 6.23, and we have the following result :

**Proposition 6.26.** *Let $k_0 \geq 1$, and $V \hookrightarrow T_e G^{k_0+2}$ such that the hypotheses for the (GGA) framework are satisfied. Let $\varphi_0 \in G^{k_0}$ and $p_0^H = (p_0^\varphi, p_0^\omega) \in T^*_{i^H(\varphi_0)} H^{k_0}$ (resp. $p_0^N = (p_0^\varphi, p_0^A) \in T^*_{i^N(\varphi_0)} N^{k_0}$) such that $p_0^G = (Ti_H)^* p_0^H$ (resp. $p_0^G = (Ti_N)^* p_0^N$). Then we have:*

1. *The normal geodesic in $H^{k_0}$ (resp. $N^{k_0}$) associated with the Hamiltonian equations with initial value $(i_H(\varphi_0), p_0^H)$ (resp. $(i_N(\varphi_0), p_0^N)$) stays in $i_H(G^{k_0})$ (resp. in $i_G(G^{k_0})$) and is a normal geodesic in the group $G^{k_0}$ starting from $(\varphi_0, p_0^G)$.*

2. *Furthermore, we have the following relation*

$$p_t^G = (Ti_H)^* p_t^H \quad \textit{(resp. } p_t^G = (Ti_N)^* p_t^N) \tag{6.9}$$

*Proof.* This is an immediate consequence of Corollary 6.23. We just need to prove the relations between the covectors. Let $(\varphi_t, p_t^G)$ and $(h_t, p_t^H)$ the corresponding Hamiltonian geodesics such that

$$m_G(\varphi_t, p_t^G) = m_H(h_t, p_t^H)$$

and $h_t = \varphi_t \cdot (\varphi_0, |J\varphi_0|) = i_H(\varphi_t)$, $p_t^H = (p_t^\varphi, p_t^\omega)$. Then for $\delta\varphi \in T_{\varphi_t} G^{k_0}$, we get

$$\begin{aligned}\left(p_t^G \mid \delta\varphi\right) &= \left(m_G(\varphi_t, p_t^G) \mid \delta\varphi \circ \varphi_t^{-1}\right) = \left(m_H(h_t, p_t^H) \mid \delta\varphi \circ \varphi_t^{-1}\right) \\ &= \left(p_t^H \mid \delta\varphi \circ \varphi_t^{-1} \cdot i_H(\varphi_t)\right) = \left(p_t^H \mid Ti_H(\delta\varphi)\right)\end{aligned}$$

which gives the equality $p_t^G = (Ti_H)^* p_t^H$ □

### 6.5.2 Regularity of the covector

In this section we arguments the introduction of the groups $H^{k_0}$ and $N^{k_0}$ by proving results on the regularity of the momentum. We first start with this general statement where we consider the Hamiltonian dual variable $p_t$ on $\mathcal{Q}$ where $\mathcal{Q} = G^{k_0}$ or $H^{k_0}$, or $N^{k_0}$.

**Proposition 6.27** (Conservation of the regularity of the covector)**.** *Let $(q_t, p_t)$ be a normal geodesic, and $u_t = \xi_{q_t}^{-1} \dot{q}_t$ its infinitesimal minimal lift. If the covector $p_t$ is $L^1$ at time $t = 1$, then it stays $L^1$ for all previous times $t \leq 1$ and given by*

$$p_t = p_1 - \int_t^1 \partial_q(\xi_{q_s}(u_s))^* p_s ds \tag{6.10}$$

*Proof.* The proof is done in the case where $\mathcal{Q} = G^{k_0} = \mathrm{Diff}_{C_0^{k_0}}(\mathbb{R}^d)$, the other cases are similar. We recall $p_t$ verifies the Hamiltonian equation with endpoint $p_1 \in L^1(\mathbb{R}^d, \mathbb{R}^d)$, i.e. $p_t$ is solution of the Linear Cauchy problem :

$$\begin{cases} \dot{p}(t) = -\partial_q(\xi_{q_t}(u_t))^* p(t) \\ p(1) = p_1 \end{cases}$$

which immediately implies we have :

$$p_t = p_1 - \int_t^1 \partial_q(\xi_{q_s}(u_s))^* p_s ds$$

Let $q \in G^{k_0}$, $u \in T_e G^{k_0+2} = C_0^{k_0+2}(\mathbb{R}^d, \mathbb{R}^d)$, the linear operator $p \in L^1 \mapsto \partial_q(\xi_q(u))^* p$ is bounded with image in $L^1$. Indeed, for $\delta q \in T_q \mathcal{Q} = C_0^{k_0}(\mathbb{R}^d, \mathbb{R}^d)$, we have :

$$\begin{aligned}(\partial_q(\xi_q(u))^* p \mid \delta q) &= (p \mid \partial_q(\xi_q(u))\delta q) \\ &= \int_{\mathbb{R}^d} \langle p(x), \partial_q(\xi_q(u))\delta q \rangle_{\mathbb{R}^d} dx \\ &= \int_{\mathbb{R}^d} \langle p(x), du(q(x))\delta q(x) \rangle_{\mathbb{R}^d} dx \\ &= \int_{\mathbb{R}^d} \langle du(q(x))^T p(x), \delta q(x) \rangle_{\mathbb{R}^d} dx\end{aligned}$$

Since $du \in C_0^{k_0+1}(\mathbb{R}^d, \mathbb{R}^{d\times d})$, therefore $(du \circ q)^T p$ is still in $L^1$, we have :

$$\begin{aligned}|\partial_q(\xi_q(u))^* p|_{L^1} &\leq |du|_{C_0^{k+1}} |p|_{L^1} \\ &\leq \mathrm{Cte}\, |u|_V |p|_{L^1}\end{aligned}$$

and the mapping $(u, q) \mapsto \partial_q(\xi_q(u))^* = (du \circ q)^T \in L(L^1(\mathbb{R}^d, \mathbb{R}^d), L^1(\mathbb{R}^d, \mathbb{R}^d))$ is continuous, and therefore the covector $p_t$ is $L^1$ for all $t \in [0, 1]$. $\square$

**Remark 6.28.** *We can adapt the previous proof and find that if $p_1$ is in $L^p$, for $p \geq 1$ (resp. in $C_0^l(\mathbb{R}^d, \mathbb{R}^d)$ for $l \leq k$), then $p_t$ stays in $L^p$ (resp. in $C_0^l(\mathbb{R}^d, \mathbb{R}^d)$) for all time.*

In the following, we will only consider the groups $G^{k_0}$ and $H^{k_0}$. We saw that when we solve the Hamiltonian equations in both $G^{k_0}$ and $H^{k_0}$, we get the following equality between the momenta

$$p_t^G = (Ti_H)^* p_t^H \tag{6.11}$$

The embedding $i_H : G^{k_0} \to H^{k_0}$ is highly non surjective (the image in $H^{k_0}$ is not even dense), and therefore the dual mapping $(Ti_H)^* : T^*H^{k_0} \to T^*G^{k_0}$ is not injective.

Let's compute the dual mapping. Let $p^G \in T^*G^{k_0}$, $p^H = (p^\varphi, p^w) \in T^*H^{k_0}$, and $\delta\varphi \in TG^{k_0}$ we have $\big((Ti_H)^* p^H \mid \delta\varphi\big) = (p^\varphi \mid \delta\varphi) + (p^\omega \mid \mathrm{div}(\delta\varphi \circ \varphi^{-1}) \circ \varphi |J\varphi|)$, i.e. we have the following (in the weak sense) :

$$(Ti_H)^* p^H = p^\varphi - |J\varphi| \left(d\varphi^*\right)^{-1} \nabla p^\omega \tag{6.12}$$

In particular, if we have $p^G = (Ti_H)^* p^H$, then we get

$$p^G = p^\varphi - |J\varphi| \left(d\varphi^*\right)^{-1} \nabla p^\omega \tag{6.13}$$

Obviously, the covector $(p^G, 0) \in T^*H^{k_0}$ satisfies equation (6.13). The non-injectivity of $(Ti_H)^*$ allows us to also consider $(p^\varphi, p^\omega)$ to gain regularity. For example, we know that if $\Omega$ is a bounded regular openset of $\mathbb{R}^d$ and $p \in L^2(\Omega)$ with $\mathrm{curl}(p) = 0$, generalisation of the Poincaré's lemma [55, 38] gives existence of $h \in H_0^1(\Omega)$ such that $\nabla h = p$. This suggest a corresponding gain of regularity in the case $\varphi = e_G$, between $p^G$ and $p^\omega$ solutions of (6.13).

In the inexact matching setting, we recall that the regularity of the covector is given by the data attachment term. Suppose that the data attachment term is given by the $C^1$ mapping $\mathcal{D} : (\varphi, \omega) \in H^{k_0} \mapsto \mathcal{D}(\varphi, \omega) \in \mathbb{R}$, and let $\varphi_1 \in G^{k_0}$ be the endpoint for the geodesic equation. Then the geodesics momenta for Hamiltonian in both $G^{k_0}$ and $H^{k_0}$ have endpoints given by

$$\begin{cases} p_1^\varphi = \partial_\varphi \mathcal{D}(\varphi_1, d\varphi_1) \\ p_1^\omega = \partial_\omega \mathcal{D}(\varphi_1, d\varphi_1) \\ p_1^G = (Ti_H)^* d_{i_H(\varphi_1)}\mathcal{D} = d_{(\varphi_1, |J\varphi_1|)}\mathcal{D} \circ T_{\varphi_1} i_H = p_1^\varphi - |J\varphi_1| \left(d\varphi_1^*\right)^{-1} \nabla p_1^\omega \end{cases}$$

Therefore if $d\mathcal{D}(\varphi_1, \omega_1) \in T^*H^{k_0}$ is $L^1$, the couple $(p_1^\varphi, p_1^\omega)$ is also $L^1$. However we will see in next subsection a case where $p_1^G$ (and therefore $p_t^G$) has singular parts coming from singularities from the target. This means the topological and geometric properties of targets and sources leads to the use of either groups $G^{k_0}$ or $H^{k_0}$.

### 6.5.3 An application to transport on atlases

We here give a motivation for using groups $H^{k_0}$ and $N^{k_0}$ to compute geodesics induced by the diffeomorphism actions. This will have particular applications in the special case of atlas (at a tissue scale for example), where the data is given as an image defined on a set of regions with discontinuities at the boundaries. In the following, we will consider a metric induced by a Hilbert space $V \hookrightarrow C_0^{k_0+2}(\mathbb{R}^d, \mathbb{R}^d)$.

Let $X_1, X_2, \ldots, X_n \subset \mathbb{R}^d$ open connected pairwise disjoint subsets of $\mathbb{R}^d$ with piecewise $C^1$ boundaries such that $\mathbb{R}^d = \cup_{i=1}^n \overline{X}_i$. We denote $\Sigma = \cup_i \partial X_i$ the boundary. We consider a template image $I_0 : \mathbb{R}^d \to \mathbb{R}$ such that $I_{0|X_i}$ is smooth for each $X_i$, and the sets $I_0(X_i)$ are pairwise disjoint, and a smooth target image $I_1 : \mathbb{R}^d \to \mathbb{R}$. We assume that both $I_0$ and $I_1$ are square integrable.

We derive the Hamiltonian equations in the space $G^{k_0}$ :

$$\begin{cases} \dot{\varphi}_t = u_t \circ \varphi_t \\ \dot{p}_t = -du_t^* \circ \varphi_t \, p_t \\ \langle u_t, v \rangle_V = (p_t \mid v \circ \varphi_t) \end{cases} \tag{6.14}$$

Now we also need to add a data attachment term to perform inexact matching. We define, for $I$ an image, the data attachment term

$$\mathcal{D}(I) = \int_{\mathbb{R}^d} (I_1 - I)^2 (y) dy = |I_1 - I|_{L^2}^2$$

or equivalently

$$\mathcal{D}(\varphi) = \mathcal{D}(\varphi \cdot I_0) = \int_{\mathbb{R}^d} \left(I_1 - I_0 \circ \varphi^{-1}\right)^2 (y) dy = \int_{\mathbb{R}^d} (I_1 \circ \varphi - I_0)^2 (x) \, |J\varphi(x)| \, dx$$

with $\varphi \in G^{k_0}$.

**Proposition 6.29.** *The data attachment term $\mathcal{D}$ is $C^1$ and the co-vector $p_1^{G^{k_0}}$ at time 1 is given as a mixture*

$$p_1 = -2 \left(I_0 - I_1 \circ \varphi\right) \nabla(\varphi_1 \cdot I_0) \circ \varphi_1 |J\varphi_1| \lambda_d + \mathcal{J} \circ \varphi_1 \left| J\varphi_{1|\Sigma} \right| \mathcal{H}_\Sigma^{d-1} \tag{6.15}$$

*where $\mathcal{H}_{|\Sigma}^{d-1}$ the $d-1$ Hausdorff measure on the boundary, and, for $x \in \varphi(\Sigma)$, $|J\varphi_{1|\Sigma}|$ is the Jacobian of $\varphi_1$ restricted to the tangent bundle of $\Sigma$ and*

$$\mathcal{J}(x) = \lim_{\epsilon \to 0} \left( \Big(I_1(x) - \varphi \cdot I_0(x + \epsilon \vec{n}(x))\Big)^2 - \Big(I_1(x) - \varphi \cdot I_0(x - \epsilon \vec{n}(x))\Big)^2 \right) \vec{n}(x)$$

*with $\vec{n}(x)$ unit normal vector on $\varphi(\Sigma)$.*

*Proof.* We first start by proving $\mathcal{D}$ is $C^1$ and we compute its derivative $d\mathcal{D}$. Let $(\varphi, \delta\varphi) \in TG^{k_0} \simeq \mathrm{Diff}_{C_0^{k_0}}(\mathbb{R}^d) \times C_0^{k_0}(\mathbb{R}^d)$. Since $I_1$ is smooth, we can differentiate under the integral sign and we have

$$
\begin{aligned}
d\mathcal{D}(\varphi)\delta\varphi = & \int_{\mathbb{R}^d} 2\left(I_1 \circ \varphi - I_0\right)(x) \left\langle \nabla I_1 \circ \varphi, \delta\varphi \right\rangle (x) \left| J\varphi(x) \right| dx \\
& + \int_{\mathbb{R}^d} \left(I_1 \circ \varphi - I_0\right)^2 (x) \left| J\varphi \right| (x) \operatorname{div}(\delta\varphi \circ \varphi^{-1}) \circ \varphi(x) dx \\
= & \int_{\mathbb{R}^d} 2\left(I_1 - \varphi \cdot I_0\right)(y) \left\langle \nabla I_1, \delta\varphi \circ \varphi^{-1} \right\rangle (y) dy \\
& + \int_{\mathbb{R}^d} \left| I_1 - \varphi \cdot I_0 \right|^2 (y) \operatorname{div}(\delta\varphi \circ \varphi^{-1})(y) dy
\end{aligned}
$$

Let's compute the second integral using Stokes theorem. We have :

$$
\begin{aligned}
& \int_{\mathbb{R}^d} \left| I_1 - \varphi \cdot I_0 \right|^2 (y) \operatorname{div}(\delta\varphi \circ \varphi^{-1})(y) dy \\
& = \sum_{i=1}^{n} \int_{\varphi(X_i)} \left| I_1 - \varphi \cdot I_0 \right|^2 (y) \operatorname{div}(\delta\varphi \circ \varphi^{-1})(y) dy \\
& = \sum_{i=1}^{n} \Bigg( -2 \int_{\varphi(X_i)} \left(I_1 - \varphi \cdot I_0\right)(y) \left\langle \nabla \left(I_1 - \varphi \cdot I_0\right), \delta\varphi \circ \varphi^{-1} \right\rangle (y) dy \\
& + \int_{\partial\varphi(X_i)} \left| I_1(y) - \lim_{\epsilon \to 0} \varphi \cdot I_0(y - \epsilon \vec{n_i}(y)) \right|^2 \left\langle \delta\varphi \circ \varphi^{-1}, \vec{n_i} \right\rangle (y) d\mathcal{H}^{d-1}_{\partial\varphi(X_i)}(y) \Bigg) \\
& = -2 \int_{\mathbb{R}^d} \left(I_1 - \varphi \cdot I_0\right) \left\langle \nabla \left(I_1 - \varphi \cdot I_0\right), \delta\varphi \circ \varphi^{-1} \right\rangle (y) dy \\
& - \int_{\varphi(\Sigma)} \left\langle \mathcal{J}, \delta\varphi \circ \varphi^{-1} \right\rangle (y) d\mathcal{H}^{d-1}_{\varphi(\Sigma)}(y)
\end{aligned}
$$

Now as $p_1 = -d\mathcal{D}(\varphi_1)$, we get that :

$$
p_1 = -2\left(I_1 \circ \varphi - I_0\right) \nabla(\varphi \cdot I_0) \circ \varphi |J\varphi| \lambda_d + \mathcal{J} \circ \varphi \left| J\varphi_{|\Sigma} \right| \mathcal{H}^{d-1}_{\Sigma} . \tag{6.16}
$$

□

In this particular case, and since the regularity of the covector is conserved, we see that this parametrization with the group of diffeomorphisms will give non-regular covectors with singularities aroung the boundarys of the atlas. This can pose problems in particular in the implementation and algorithms as it can lead to stability issues. However, using the over-parametrization given by the group $H^k$ we can rewrite the data attachment term by

$$
\mathcal{D}(\varphi) = \tilde{\mathcal{D}}(\varphi, |J\varphi|),
$$

where

$$
\tilde{\mathcal{D}}(\varphi, \omega) = \int_{\mathbb{R}^d} \left(I_1 \circ \varphi(x) - I_0(x)\right)^2 \omega(x) dx
$$

for any $(\varphi, \omega) \in H^k$. The corresponding covectors $(p_1^\varphi, p_1^\omega)$ at time 1 then become

$$
\begin{cases}
p_1^\varphi = -\partial_\varphi \tilde{\mathcal{D}}(\varphi_1, |J\varphi_1|) = 2\left(I_0 - I_1 \circ \varphi\right) |J\varphi_1| \nabla I_1 \circ \varphi \\
p_1^\omega = -\partial_\omega \tilde{\mathcal{D}}(\varphi_1, |J\varphi_1|) = -|I_0 - I_1 \circ \varphi|^2
\end{cases}
$$

which is therefore $L^1$ in space. Since the regularity is conserved (cf. proposition 6.27) the covectors stay $L^1$ for all time $t$. In practice, since we often use over-parametrization, this means we could trade off memory for stability, because we would gain here some regularity for the covectors.

# Chapter 7

# Application to multiscale matching

## Contents



We propose here applications of the previous chapters to multiscale matching using the LDDMM framework. The main idea is to consider simultaneous actions of a coarse-to-fine stacks of transformations acting on a shape space describing a shape at different resolutions with interaction across scales. Such approaches were first described in [18, 84, 89], where authors used a sum of kernels allowing to produce several diffeomorphisms for each scale, and then later in [71] where authors directly describe general sub-Riemannian metrics on a product space of diffeomorphisms groups. We adapt these ideas in this chapter.

We start in section 7.1 by recalling the definition of reproducible kernel Hilbert spaces, which provide very useful Hilbert spaces of vector fields. This allows to equip the group of diffeomorphisms with right-invariant sub-Riemannian structures that are easy to use computationally. In particular, we develop the induced metric on the space of landmarks for the simple action of transport of points by diffeomorphisms, recovering classical formulation of the Large deformation diffeomorphic metric mapping (LDDMM) framework. This constitutes a first brick for the next sections.

Section 7.2 describes the main results of the first part of [74] and is a joint work with PhD student Rayane Mouhli. Here we only deal with two scales, where the group of deformation representing the coarsest scale is a finite dimensional Lie group (for example rotations, isometries, scalings, etc.) and the finest scale is the group of $C_0^k$ diffeomorphisms equipped with a right-invariant metric given by a RKHS. We describe how we can construct then a new group of deformations and propose several applications to matching of curves, landmarks or images.

Section 7.3 describes results from [82, Section 4] and is devoted to the case of multi-scale shape spaces for registration through the action of the product of diffeomorphisms following [71]. We also describe how this setting can be coupled with a rigid matching.

## 7.1 Reproducible kernel Hilbert spaces

### 7.1.1 General definition

The results of chapter 5 and their applications to shape registration in 6 relies on the choice of a Hilbert space $V$ to construct right-invariant sub-Riemannian metrics. In most examples, the action of diffeomorphisms is an important part of the global deformation applied to the source shape, and therefore one must choose carefully a Hilbert space of vector fields. A useful class of such Hilbert spaces is the one of reproducible kernel Hilbert spaces (or simply RKHS) [11], because they provide easily computable vector fields for the minimal lift when the shape is given by a set of landmarks [98], which is an important special case since it includes most practical situations after discretization. In this part, we briefly recall the definition and some properties of the RKHS (see. [11] for more details).

**Definition 7.1** (Reproducible kernel Hilbert space)**.** A Hilbert space $(V, \langle \cdot, \cdot \rangle_V)$ of maps of $\mathbb{R}^d$ is said to be a *reproducible kernel Hilbert space* if, for any $x \in \mathbb{R}^d$, the evaluation $\delta_x : f \in V \mapsto f(x)$ is continuous.

Let $V$ be a RKHS, and denote by $K_V : V^* \to V$ its Riezs isometry. In such case, for any $(x, b) \in (\mathbb{R}^d)^2$, the map $\delta_x^b : V \to \mathbb{R}$ defined by

$$(\delta_x^b \mid f) = b^\top f(x)$$

is continuous, i.e. belongs to $V^*$. This allows to define a map $k_V : (\mathbb{R}^d)^2 \to \mathbb{R}^{d \times d}$, called the *reproducible kernel* of $V$, such that, for any $x, y, b \in \mathbb{R}^d$

$$k_V(x, \cdot)b = K_V \delta_x^b$$

It is easy to see that for any $x, y, a, b \in \mathbb{R}^d$, we also have

$$\langle k_V(x, \cdot)a, k_V(y, \cdot)b \rangle_V = a^\top k_V(x, y)b$$

so that, in particular $k_V(x, y) = k_V(y, x)^\top$, and $k_V(x, x)$ is symmetric positive. Moreover, the interest is that the kernel determines the Hilbert space, meaning that we can just define the kernel $k_V$ and not the total Hilbert space $V$ [11]. Moreover, the regularity of the kernel gives informations on the Hilbert space $V$. Let $l \geq 0$, we say that $V$ is of class $C_0^l$ if it is continuously embedded in $C_0^l(\mathbb{R}^d, \mathbb{R}^d)$. In particular, we have the following:

**Proposition 7.2** (RKHS of class $C_0^l$)**.** *Let $V$ be a RKHS and $l \geq 0$. Then $V$ is of class $C_0^l$ if and only if its reproducible kernel $k_V$ is $C_0^{2l}$.*

**Example 7.3** (Gaussian kernel). *We present here the Gaussian kernel, which is the main example of kernel that we are going to use in this chapter for applications. Let $\sigma > 0$. It represents the scale at which we perform the deformations. We then define the smooth Gaussian kernel $k_\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ given by*

$$k_\sigma(x, y) = \exp(-\frac{|x-y|^2}{2\sigma^2})Id$$

*and denote by $V_\sigma$ the associated reproducible kernel space. Note that the kernel is of the form $\gamma(|x-y|)Id$ where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a scalar map, and in particular this implies that rotations and translations are isometries of $V_\sigma$, in the sense that for any $(R, \tau) \in SO_d \times \mathbb{R}^d$ and $u, v \in V_\sigma$, we get*

$$\left\langle R^\top u(R\cdot+\tau), R^\top v(R\cdot+\tau)\right\rangle_{V_\sigma} = \langle u, v\rangle_{V_\sigma}.$$

### 7.1.2 Application to LDDMM and transport of landmarks

We recall in this section an application to classic large deformation model, using a Gaussian kernel for registration of landmarks. Let $n \geq 1$, and

$$\mathcal{Q} = \mathrm{Lmk}_n(\mathbb{R}^d) = \{\mathbf{q} = (q_i) \in (\mathbb{R}^d)^n \mid q_i \neq q_j \text{ if } i \neq j\}$$

be the space of $n$-landmarks. The group of diffeomorphisms $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ acts on $\mathcal{Q}$ by

$$\varphi \cdot (q_i)_{i\leq n} = (\varphi(q_i))_{i\leq n}$$

and the induced infinitesimal action becomes $\xi_{\mathbf{q}} u = (u(q_i))_{i\leq n}$ for $\mathbf{q} = (q_i)_i$. This action satisfies (S.1-4) 6.1. Let $V$ be a RKHS induced by a gaussian kernel $k_\sigma$ with scale $\sigma > 0$, so that we can define a right-invariant structure on the group $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ (cf. 5.1), and which induces a metric on $\mathcal{Q}$ following 6.2. In such case the horizontal subspace $\Delta_{\boldsymbol{q}} := \xi_{\boldsymbol{q}}(V_\sigma) \subset T_{\boldsymbol{q}}\mathcal{Q}$ is then exactly equal to the tangent space $T_{\mathbf{q}}\mathcal{Q}$, and we recall that, for any $\boldsymbol{q} = (q_i)_{i\leq n}$, the induced metric on $T_{\boldsymbol{q}}\mathcal{Q}$ then is given by

$$\forall \delta\boldsymbol{q} \in T_{\boldsymbol{q}}\mathcal{Q}, \quad \langle \delta\boldsymbol{q}, \delta\boldsymbol{q}\rangle_{\boldsymbol{q}} = \inf_{u\in V_\sigma,\ \xi_{\boldsymbol{q}}u=\delta\boldsymbol{q}} \langle u, u\rangle_{V_\sigma} = \langle \xi_{\boldsymbol{q}}^{-1}\delta\boldsymbol{q}, \xi_{\boldsymbol{q}}^{-1}\delta\boldsymbol{q}\rangle_{V_\sigma}$$

where $\xi_{\boldsymbol{q}}^{-1}$ is the inverse of the restriction $\xi_{\boldsymbol{q}} : (\ker \xi_{\boldsymbol{q}})^\perp \to T_{\boldsymbol{q}}\mathcal{Q}$. This means we obtain a Riemannian metric on $\mathcal{Q}$, and we recall the following well-known reduction principle [98]:

**Proposition 7.4** (Reduction principle in LDDMM). *Let $\delta\boldsymbol{q} \in T_{\boldsymbol{q}}\mathcal{Q}$. Then there exists $\boldsymbol{p} = (p_i)_{i\leq n} \in (\mathbb{R}^d)^n$ such that the minimal lift $u = \xi_{\boldsymbol{q}}^{-1}\delta\boldsymbol{q}$ of $\delta\boldsymbol{q}$ is given by*

$$u = K_V \sum_{i\leq n} \delta_{q_i}^{p_i} = \sum_{i\leq n} k_\sigma(q_i, \cdot)p_i. \tag{7.1}$$

*Moreover the norm of $u$ is therefore*

$$|u|_{V_\sigma}^2 = \sum_{i,j\leq n} k_\sigma(q_i, q_j)\langle p_i, p_j\rangle$$

*Proof.* This is a direct consequence of proposition 6.5 since $\Delta_q = T_q\mathcal{Q}$ is therefore closed in $T_q\mathcal{Q}$. □

This is also consistent with the Hamiltonian formulation for the normal geodesics (cf. 6.3). Indeed we also obtain a normal Hamiltonian $H$ defined on $T^*\mathcal{Q}$ by

$$H(\boldsymbol{q},\boldsymbol{p}) = \sum_{i,j} k_\sigma(q_i,q_j)\langle p_i,p_j\rangle.$$

This permits a parametrization of vector fields by a finite number of parameters $p_1,\dots p_n$, and the matching problem between a source object $\boldsymbol{q_S}\in\mathcal{Q}$ and a target $\boldsymbol{q}_T\in\mathcal{Q}$ becomes

$$\inf_{\boldsymbol{p}=(p_1,\dots p_n)\in(\mathbb{R}^d)^n} J(p_1,\dots p_n) = \frac{1}{2}\int_0^1 \sum_{i,j\le n} k_\sigma(q_{i,t},q_{j,t})\langle p_{i,t},p_{j,t}\rangle\,dt + \mathcal{D}(\boldsymbol{q}_1) \qquad (7.2)$$
$$\text{s.t}\quad \begin{cases} (\dot{\boldsymbol{q}}_t,\dot{\boldsymbol{p}}_t) = \nabla^\omega H(\boldsymbol{q}_t,\boldsymbol{p}_t) \\ (\boldsymbol{q}_0,\boldsymbol{p}_0) = (\boldsymbol{q_S},\boldsymbol{p}) \end{cases}$$

## 7.2 Large deformation model coupled with the action of a finite dimensional Lie group

This section is taken from [74]. We introduce here the general theoretical framework for coupling the classic large deformation model [17] with the action of a finite-dimensional Lie group. Such a construction will serve as foundation for the applications presented in the subsequents sections. The differential structure and the regularity properties of the group of deformations considered in this section is presented in 7.2.1. We introduce a right-invariant sub-Riemannian metric on this group and recall the classical completeness results. Following chapter 6, we then specify in section 7.2.2 the hypotheses required on the differents actions and spaces to induce a sub-Riemannian structure on the space of shapes. This will serve to define a distance between shapes that will be used as a metric to introduce a variational problem for the matching of shapes, taking into account the deformations of the group.

### 7.2.1 A semidirect product of diffeomorphisms and a finite dimensional Lie group

In classical large deformation model [93, 17, 6], shapes deformation are performed through an action of the group of diffeomorphisms with finite regularity ($C^k$ or Sobolev). Following Lie group ideas, small deformations can be described as vector fields on $\mathbb{R}^d$, representing at each point the directions and speed guiding the evolution of shapes. This group of diffeomorphisms can be enriched by another finite-dimensional group $G$ that can also act on shapes, to define constrained deformations as rotations, translations or scalings. We define the group of deformations as a semidirect product of $G$ with the group of diffeomorphisms.

#### 7.2.1.1 First definitions and differentiable structure

Let $G$ be a finite-dimensional Lie group and $\mathfrak{g}$ be its Lie algebra. The purpose of this part is to establish the theoretical framework for coupling the Lie group $G$ with the group of $C^k$-diffeomorphisms

$$\mathrm{Diff}_{C_0^k}(\mathbb{R}^d) = \big(\mathrm{id} + C_0^k(\mathbb{R}^d,\mathbb{R}^d)\big)\cap \mathrm{Diff}^1(\mathbb{R}^d)$$

We will first make some assumptions to define this coupling as a semidirect product of $G$ and $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ along with its differentiable structure compatible with the composition law. We consider the following hypotheses:

**Action of $G$ on $\mathbb{R}^d$** We suppose $G$ acts smoothly via diffeomorphisms on $\mathbb{R}^d$ and we denote $g \cdot x$ the action of $g \in G$ on $x \in \mathbb{R}^d$.

**Action of $G$ on $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$** We suppose that the action of $G$ on $\mathbb{R}^d$ can be lifted to a continuous right action through smooth automorphisms on $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$. We denote by $\rho_g(\varphi)$ the action of $g \in G$ on $\varphi \in \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$. This means that we have a continuous mapping $\rho : G \to \mathrm{Aut}(\mathrm{Diff}_{C_0^k}(\mathbb{R}^d))$, such that for all $\varphi, \psi \in \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ and $g, h \in G$, we have

$$\rho_{gh}(\varphi) = \rho_h(\rho_g(\varphi)), \tag{7.3}$$
$$\rho_g(\varphi \circ \psi) = \rho_g(\varphi) \circ \rho_g(\psi), \tag{7.4}$$
$$g^{-1} \cdot \varphi(g \cdot x) = \rho_g(\varphi)(x), \quad \forall x \in \mathbb{R}^d \tag{7.5}$$

and that the mapping $\varphi \mapsto \rho_g(\varphi)$ is a smooth diffeomorphism on $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$. The third condition (7.5) is the compatibility condition between the action of $G$ on $\mathbb{R}^d$ and the action of $G$ on $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$.

**Regularity of the action** For $u \in T_{\mathrm{id}} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) = C_0^k(\mathbb{R}^d, \mathbb{R}^d)$, since the action $\rho_g$ is differentiable, we can consider $T_{\mathrm{id}}\rho_g u = \partial_\varphi|_{\varphi=\mathrm{id}}\rho_g(\varphi) \cdot u$ the infinitesimal action of $g$ on $u$. Finally, we suppose that the action $\rho$ induces a continuous action on $\mathrm{Diff}_{C_0^{k+l}}(\mathbb{R}^d)$ by smooth diffeomorphisms, and such that

$$\begin{array}{ccc} G \times \mathrm{Diff}_{C_0^{k+l}}(\mathbb{R}^d) & \longrightarrow & \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) \\ g, \varphi & \longmapsto & \rho_g(\varphi) \end{array}$$

is $C^l$.

Under those assumptions, we can define the semidirect product :

$$\mathcal{G}^k = G \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$$

with composition law

$$(g, \varphi)(g', \varphi') = (gg', \rho_{g'}(\varphi) \circ \varphi') \tag{7.6}$$

**Remark 7.5** (Inclusion in $\mathrm{Diff}_{C^k}(\mathbb{R}^d)$)**.** *The conditions we stated before are quite natural since the semidirect product $\mathcal{G}^k$ is a subgroup of the group $\mathrm{Diff}_{C^k}(\mathbb{R}^d)$ of all $C^k$ diffeomorphisms of $\mathbb{R}^d$. Indeed, the first hypothesis of the action of $G$ on $\mathbb{R}^d$ can be restated as a group inclusion*

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Diff}_{C^k}(\mathbb{R}^d) \\ g & \longmapsto & [\varphi_g : x \mapsto g \cdot x] \end{array}$$

*This allows to define the injection*

$$i : \left\{ \begin{array}{ccc} \mathcal{G}^k & \longrightarrow & \mathrm{Diff}_{C^k}(\mathbb{R}^d) \\ g, \varphi & \longmapsto & [\varphi_g \circ \varphi : x \mapsto g \cdot \varphi(x)] \end{array} \right.$$

*Note that this injection is also a group morphism, since*

$$\begin{aligned} i\big((g,\varphi)(g',\varphi')\big)(x) &= i(gg',\rho_{g'}(\varphi)\circ\varphi')(x) \\ &= gg'\cdot(\rho_{g'}(\varphi)\circ\varphi'(x)) \\ &= g\cdot\varphi(g'\cdot\varphi'(x)) \\ &= i(g,\varphi)\circ i(g',\varphi')(x). \end{aligned}$$

*In the next sections, we will define how the group $\mathcal{G}^k$ acts on shape spaces. In particular these actions will be compatible with the action of the bigger group $\mathrm{Diff}_{C^k}(\mathbb{R}^d)$ on shape spaces. However this description as a semidirect product allows to separated both the action of $G$ and of the group of diffeomorphisms $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$.*

Using the regularity conditions of the action of $G$ on the group $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$, we prove that the group $\mathcal{G}^k$ is a Banach half-Lie group, and some regularity properties of the multiplication and the inverse in the group.

**Proposition 7.6** (Differential Structure of $\mathcal{G}^k$)**.** *The group $\mathcal{G}^k$ is a Banach right half-Lie group. Furthermore the subgroup $\mathcal{G}^{k+l}$ is exactly the group of $C^l$-differentiable elements of $\mathcal{G}^k$, and the family $\{\mathcal{G}^k, k\geq 1\}$ is an admissible graded group structure satisfying 4.3.*

*Proof.* The group $\mathcal{G}^k$ is immediately a topological group and a Banach manifold for $k\geq 1$. It remains to prove that right translations in the group are smooth. Since $G$ is a Lie group, it is sufficient to prove that for any $(h,\psi)\in G\ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d,\mathbb{R}^d)$, the mapping

$$\begin{array}{ccc} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) & \longrightarrow & \mathrm{Diff}_{C_0^k}(\mathbb{R}^d) \\ \varphi & \longmapsto & \rho_h(\varphi)\circ\psi \end{array}$$

is smooth. This follows from the fact that $\rho$ is smooth and right translations in $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ are smooth since it is a half-Lie group. Therefore the group $\mathcal{G}^k$ is a Banach half-Lie group. We prove now the properties (G.1-5). Moreover, the group $\mathrm{Diff}_{C_0^{k+l}}(\mathbb{R}^d)$ is a subgroup of $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ with smooth inclusion, and thus (G.1) follows. We then prove that the subgroup

$$(\mathcal{G}^k)^l = \{(g,\varphi)\in\mathcal{G}^k,\ L_{(g,\varphi)} \text{ is } C^l\}$$

of $C^l$-differentiable elements of $\mathcal{G}^k$ as defined in [16] is exactly $\mathcal{G}^{k+l}$. Since $G$ is a Lie group (and therefore $G^l=G$), and the space of $C^l$-elements of $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ is $\mathrm{Diff}_{C_0^{k+l}}(\mathbb{R}^d)$ [16], then the space $(\mathcal{G}^k)^l$ is simply

$$(\mathcal{G}^k)^l = \{(g,\varphi)\in\mathcal{G}^k,\ \text{s.t. } \forall h\in G,\ \rho_h(\varphi)\in \mathrm{Diff}_{C_0^{k+l}}(\mathbb{R}^d)\}.$$

Therefore if $(g,\varphi)\in(\mathcal{G}^k)^l$, by taking $h=e_G$, we immediately have $\varphi\in\mathrm{Diff}_{C_0^{k+l}}(\mathbb{R}^d)$. Conversely if $\varphi\in\mathrm{Diff}_{C_0^{k+l}}(\mathbb{R}^d)$, by hypothesis, $\rho_h(\varphi)$ is also in $\mathrm{Diff}_{C_0^{k+l}}$ for any $h\in G$. In particular, (G.3) follows, and by implicit function theorem, (G.2) is also proved as a consequence of (G.3). Moreover the tangent space at identity of $\mathcal{G}^k$ is given by

$$T_{(e_G,\mathrm{id})}\mathcal{G}^k = C_0^k(\mathbb{R}^d,\mathbb{R}^d)\oplus\mathfrak{g}$$

where $\mathfrak{g}$ denotes the Lie algebra of the group $G$. By smoothness of $T_{e_G}R$ in the Lie group $G$, and the differentiable conditions on the map $\rho: G\times\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)\to\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$, (G.4) also follows. Finally, for $(g,\varphi)\in\mathcal{G}^{l+1}$, the differential of the left multiplication is given by

$$T_{(e_G,\mathrm{id})}L(g,\varphi)(X,u)=\left(T_{e_G}L_g(X),d\left(\rho_g(\varphi)\right)u\right),$$

where $d\left(\rho_g(\varphi)\right)u:x\mapsto T_x(\rho_g(\varphi))u(x)$ is $C^k$, so that (G.5) is also proved. $\square$

**Remark 7.7** (Right-invariant local addition on $\mathcal{G}$)**.** *Note that if the action of $G$ on $\mathbb{R}^d$ is proper, the group $\mathcal{G}$ can be equipped with a right-invariant local addition. Indeed, since $G$ is a finite-dimensional Lie group, it can be endowed with a right-invariant Riemannian metric and therefore with a right-invariant local addition through the exponential map. We denote this local addition by $\tau_G:V_G\subset TG\to G$, where $V_G$ is an open neighborhood of the zero section of $TG$. Moreover, since the group $G$ acts properly on $\mathbb{R}^d$, there exists a Riemannian metric on $\mathbb{R}^d$ that is $G$-invariant [58, theorem 2]. Therefore the exponential map* exp *of this metric is a local addition on $\mathbb{R}^d$ that is equivariant with regards to the action of $G$, i.e. for $g\in G,(x,v)\in\mathbb{R}^d\times\mathbb{R}^d$, we have*

$$\exp_{g\cdot x}(g\cdot v)=g\cdot\exp_x(v)$$

*Now following [16], this induces a right-invariant local addition $\tau_{\mathrm{Diff}}$ on $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ by taking the push-forward*

$$\exp_*:\begin{array}{ccc}T\,\mathrm{Diff}_{C_0^k}(\mathbb{R}^d) & \longrightarrow & \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)\\ (\varphi,v\circ\varphi) & \longmapsto & \exp_\varphi(u\circ\varphi)\end{array},$$

*and because of the compatibility condition* (7.5)*, this local addition is also equivariant with regards to the action of $G$. One can easily verify that the product $\tau_G\times\tau_{\mathrm{Diff}}$ thus defines a right-invariant local addition on $\mathcal{G}^k$. Therefore, under this condition, by [16, theorem 3.4], the regularity conditions (G.1-5) directly follows. However, we will give some examples of group $G$ that do not act properly on $\mathbb{R}^d$.*

We recall, following 4.3, that the groups $\mathcal{G}^{k+1}$ with $k\geq 1$ are $L^p$- regular

**Proposition 7.8** (Existence and uniqueness of a flow in $\mathcal{G}^k$)**.** *Let $(X_t,u_t)\in L^2([0,1],\mathfrak{g}\times C_0^{k+1}(\mathbb{R}^d,\mathbb{R}^d))$ be time-varying vector fields. There exists a unique solution $(g_t,\varphi_t)\in AC_{L^2}([0,1],\mathcal{G}^k)$, called the flow of $(X_t,u_t)$, to the system*

$$\begin{cases}(\dot g_t,\dot\varphi_t)=T_{(e_G,\mathrm{id})}R_{(g_t,\varphi_t)}(X_t,u_t)\\ (g_0,\varphi_0)=(e_G,\mathrm{id})\end{cases}\tag{7.7}$$

*Proof.* This is a direct consequence of proposition 4.18. $\square$

We conclude this section with some examples that will be further developed in the following sections.

**Example 7.9** (Isometries and diffeomorphisms)**.** *Consider the group* $\mathrm{Isom}(\mathbb{R}^d):=\mathbb{R}^d\ltimes\mathrm{SO}_d$ *of isometries of $\mathbb{R}^d$. It naturally acts by conjugation on the group of diffeomorphisms by*

$$((R,T)\cdot\varphi)(x)=R^\top\varphi(Rx+T)-R^\top T.$$

*This allows to define the semidirect product* $\mathrm{Isom}(\mathbb{R}^d)\ltimes\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$.

**Example 7.10** (Anistropic scalings and diffeomorphisms). *For $\rho = (\rho_1, ..., \rho_d) \in \mathbb{R}^d_{>0}$, we denote $\mathsf{D}_\rho \in M_d(\mathbb{R})$ the diagonal matrix with coefficients $(\rho_i)_i$*

$$\mathsf{D}_\rho = \begin{bmatrix} \rho_1 & 0 & \cdots & 0 \\ 0 & \rho_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_d \end{bmatrix}.$$

*The scaling group also acts by conjugation on the diffeomorphism group by*

$$(\rho \cdot \varphi)(x) = \mathsf{D}_{\rho^{-1}}\varphi(\mathsf{D}_\rho x).$$

*allowing to define the semidirect product group of anisotropic scalings and diffeomorphisms $\mathbb{R}^d_{>0} \ltimes \mathrm{Diff}_{C^k_0}(\mathbb{R}^d)$. Note that in this example, the action of the group of anisotropic scalings $\mathbb{R}^d_{>0}$ on $\mathbb{R}^d$ is not proper, since the stabilizer of $0 \in \mathbb{R}^d$*

$$\mathrm{Stab}_{\mathbb{R}^d_{>0}}(0) = \{\rho \in \mathbb{R}^d, \text{ s.t. } \rho \cdot 0 = 0\}$$

*is the whole group $\mathbb{R}^d_{>0}$, which is not compact. In particular, we cannot use the construction of 7.7.*

#### 7.2.1.2 Sub-Riemannian metric on $\mathcal{G}^k$

In this section, we introduce right-invariant metrics on the group $\mathcal{G}^k$ following the general framework in [82], [81]. Indeed, proposition 7.6 places our setting exactly within this framework, from which the completeness of these metrics follow. Suppose the algebra $\mathfrak{g}$ is equipped with a scalar product $\langle \cdot, \cdot \rangle_\mathfrak{g}$, so that we define the natural right-invariant metric :

$$\langle X, Y \rangle_g = \langle X g^{-1}, Y g^{-1} \rangle_\mathfrak{g} \tag{7.8}$$

Let also $V$ be a Hilbert space continuously embedded in $C^{k+2}_0(\mathbb{R}^d, \mathbb{R}^d) = T_{\mathrm{id}} \mathrm{Diff}_{C^{k+2}_0}(\mathbb{R}^d)$ and the map $K_V : V^* \to V$ denotes the inverse of the Riesz isometry on $V$. This defines a right-invariant sub-Riemannian structure on $\mathcal{G}^k$ defined by

- the vector bundle $\mathfrak{g} \times V$ on $\mathcal{G}^k$, that is equivalent to the bundle $\Delta_{g,\varphi} := T_g G \times (d_{\mathrm{id}}\rho_g(V)) \circ \varphi \hookrightarrow T_{g,\varphi}\mathcal{G}^k$
- the morphism $d_{e_G,\mathrm{id}} R_{g,\varphi} : \begin{cases} \mathfrak{g} \times V & \longrightarrow & \Delta_{g,\varphi} \\ X, u & \longmapsto & (d_{e_G} R_g(X), (d_{\mathrm{id}}\rho_g(u)) \circ \varphi) \end{cases}$
- the metric $\langle (X, u),\ (X, u) \rangle = \langle X, X \rangle_\mathfrak{g} + \langle u, u \rangle_V$

Let $L\left((g_t, \varphi_t), (X_t, u_t)\right)$ denotes the usual sub-Riemannian length given by

$$L\left((g_t, \varphi_t), (X_t, u_t)\right) = \int_0^1 \sqrt{|X_t|^2_\mathfrak{g} + |u_t|^2_V}\, dt$$

We can equip $\mathcal{G}^k$ with the associated sub-Riemannian distance $d_{SR}$ defined as in [8, 6, 82]

$$d_{SR}\left((g, \varphi), (g', \varphi')\right) := \inf\{L\left((g_t, \varphi_t), (X_t, u_t)\right) \mid \text{s.t. } ((g_t, \varphi_t), (X_t, u_t)) \in \mathrm{Hor}_{L^1}\left((g, \varphi), (g', \varphi')\right)\}.$$

where $\mathrm{Hor}_{L^1}\left((g, \varphi), (g', \varphi')\right)$ is the set of $L^1$ horizontal systems joining $(g, \varphi)$ to $(g', \varphi')$, i.e curves $((g_t, \varphi_t), (X_t, u_t))$ with $(X_t, u_t) \in L^1([0, 1], V)$, $(g_t, \varphi_t) \in AC_{L^1}([0, 1], \mathcal{G}^k)$ such that for all $t \in [0,1]$, $(\dot{g}_t, \dot{\varphi}_t) = d_{(e_G,\mathrm{id})} R_{(g_t,\varphi_t)}(X_t, u_t)$. Equivalently, the sub-Riemannian distance can be obtained by minimizing the energy [9, 6, 82]

$$E\left((g_t,\varphi_t),(X_t,u_t)\right) = \int_0^1 |X_t|_{\mathfrak{g}}^2 + |u_t|_V^2 \, dt.$$

We will rely on this expression to determine minimizers. The space $(\mathcal{G}^k, d_{SR})$ is therefore a metric space and we have the following completeness properties as a consequence of theorem 5.7.

**Proposition 7.11** (Completeness of $\mathcal{G}^k$). *We get the following*

1. *The space $(\mathcal{G}^k, d_{SR})$ is metrically complete.*
2. *Suppose $V$ is $G$-invariant. Then the space $(\mathcal{G}^k, d_{SR})$ is geodesically convex, meaning that for any $(g,\varphi),(g',\varphi') \in \mathcal{G}^k$ such that $d_{SR}\left((g,\varphi),(g',\varphi')\right) < \infty$, there exists a minimizing geodesic connecting $(g,\varphi)$ and $(g',\varphi')$*

*Proof.* First point is already proved in 5.7. We now prove the second point, namely that condition (G.6) (cf. 5.1.2.1) applies in this case. Since for any $g \in G$, the mapping $d_{\mathrm{id}}\rho_g$ is an isometry of $V$, then we get that for any horizontal system $((g_t,\varphi_t),(X_t,u_t))$,

$$\begin{aligned} L\left((g_t,\varphi_t),(X_t,u_t)\right) &= \int_I \sqrt{|X_t|_{\mathfrak{g}}^2 + |u_t|_V^2} dt \\ &= \int_I \sqrt{|X_t|_{\mathfrak{g}}^2 + |d_{\mathrm{id}}\rho_{g_t}(u_t)|_V^2} dt \\ &= L\left((g_t,\varphi_t),(X_t, d_{\mathrm{id}}\rho_{g_t}(u_t))\right) \end{aligned}$$

Therefore, we consider the change of variable $\tilde{u} = d_{\mathrm{id}}\rho_g(u)$ and we get that

$$d_{SR}\left((g,\varphi),(g',\varphi')\right) = \inf_{(X,\tilde{u}) \in L^2([0,1],\mathfrak{g}\times V)} L\left((g_t,\varphi_t),(X_t,u_t)\right) \tag{7.9}$$

$$\text{s.t} \quad \begin{cases} \dot{g}_t &= T_{e_G} R_{g_t}(X_t) \\ \dot{\varphi}_t &= \tilde{u}_t \circ \varphi_t \end{cases} \tag{7.10}$$

We define the endpoint mapping $\mathrm{End} : L^2(I, \mathfrak{g} \times V) \to \mathcal{G}^k$ such that for any $(X,u) \in L^2([0,1], \mathfrak{g} \times V)$, $\mathrm{End}(X,u) = (g_1, \varphi_1)$ where the curve $(g_t,\varphi_t)$ satisfies the dynamic (7.10). To prove geodesic convexity, it suffices to prove that the endpoint mapping is weakly continuous [82, 9]. This holds since $G$ is a (finite dimensional) Lie group and by [95, 93, 17] the endpoint mapping $u \in L^2(I,V) \to \varphi_1^u$ is weakly continuous, where $\varphi_t$ satisfies

$$\dot{\varphi}_t^u = u_t \circ \varphi_t^u, \quad \varphi_0^u = \mathrm{id}$$

□

Finally, following 5.2, this sub-Riemannian structure defines a co-metric on $T^*\mathcal{G}^k$ given by

$$K_{g,\varphi}(p^g, p^\varphi) = (T_{e_G} R_g K_{\mathfrak{g}} (T_{e_G} R_g)^* p^g, T_{\mathrm{id}} R_\varphi T_{\mathrm{id}} \rho_g K_V (T_{\mathrm{id}} \rho_g)^* (T_{\mathrm{id}} R_\varphi)^* p^\varphi)$$

and the corresponding Hamiltonian given by

$$\begin{aligned} H\left(g,\varphi,p^g,p^\varphi\right) &= \frac{1}{2}\left(K_{g,\varphi}(p^g,p^\varphi),(p^g,p^\varphi)\right) \\ &= \frac{1}{2}(|K_{\mathfrak{g}}(T_{e_G} R_g)^* p^g|_{\mathfrak{g}}^2 + |K_V (T_{\mathrm{id}}\rho_g)^* (T_{\mathrm{id}} R_\varphi)^* p^\varphi|_V^2) \end{aligned}$$

We recall the following result

**Proposition 7.12** (Sub-Riemannian geodesics). *Let $(g_t, \varphi_t) \in \mathrm{Hor}_{L^1}$ a horizontal system. Then $(g_t, \varphi_t, X_t, u_t)$ is a normal geodesic if and only if there exists a covector $(p^g_t, p^\varphi_t) \in T_{g_t,\varphi_t}\mathcal{G}^k$ such that*

$$(\dot{g}_t, \dot{\varphi}_t) = \nabla^\omega H(g_t, \varphi_t, p^g_t, p^\varphi_t). \tag{7.11}$$

*Proof.* This follows from proposition 5.10 □

We will mostly focus here the case where $V$ is $G$-invariant, in the sense that $T_{\mathrm{id}}\rho_g V = V$ for all $g \in G$, and that $T_{\mathrm{id}}\rho_g$ is an isometry of $V$. We will see that this property is related to whether $G$ acts properly on $\mathbb{R}^d$, since the construction of such Hilbert spaces $V$ relies on the construction of $G$-invariant metrics in $\mathbb{R}^d$. In such case, the Hamiltonian simplifies to

$$H\left(g, \varphi, p^g, p^\varphi\right) = \frac{1}{2}(|K_{\mathfrak{g}}(T_{e_G}R_g)^* p^g|^2_{\mathfrak{g}} + |K_V(T_{\mathrm{id}}R_\varphi)^* p^\varphi|^2_V).$$

### 7.2.2 Action on shape spaces

Let $\mathcal{Q}$ be a Banach manifold, representing the shape space. We define then the augmented shape space $\tilde{Q} = G \times \mathcal{Q}$, where the $G$ part gives a representation of the shape in a coarsest scale. For example, it can represent the position of the shapes in $\mathbb{R}^d$ if $G$ is the group of translations, or the orientation if $G$ is $SO_d$. The augmented shape space $\tilde{Q}$ allows to keep track of the action of $G$ on the shapes.

We suppose now that the group $\mathrm{Diff}_{C^k_0}(\mathbb{R}^d)$ acts on $\mathcal{Q}$ with following regularity conditions [5]

**Continuity of the action** The action $\varphi, q \mapsto \varphi \cdot q$ is continuous.

**Infinitesimal action** For all $q \in \mathcal{Q}$, the mapping $\varphi \mapsto \varphi \cdot q$ is $C^\infty$, and we denote for $u \in C^k_0(\mathbb{R}^d, \mathbb{R}^d)$, $u \cdot q = \partial_\varphi|_{\varphi=\mathrm{id}}(\varphi \cdot q)$ its continuous differential in id, also called the infinitesimal action of $u$ on $q$.

**Regularity of the action** For $l > 0$, the mappings

$$\begin{array}{ccc} \mathrm{Diff}_{C^{k+l}_0} \times \mathcal{Q} & \longrightarrow & \mathcal{Q} \\ (\varphi,\ q) & \longmapsto & \varphi \cdot q \end{array} \quad \text{and} \quad \begin{array}{ccc} C^{k+l}_0(\mathbb{R}^d, \mathbb{R}^d) \times \mathcal{Q} & \longrightarrow & T\mathcal{Q} \\ (u,\ q) & \longmapsto & u \cdot q \end{array}$$

are $C^l$.

Suppose $G$ also acts smoothly via diffeomorphisms on $\mathcal{Q}$, and suppose the following compatibility condition

$$\varphi \cdot (g \cdot q) = g \cdot (\rho_g(\varphi) \cdot q). \tag{7.12}$$

Therefore, we define the group action of $\mathcal{G}^k = G \ltimes \mathrm{Diff}_{C^k_0}(\mathbb{R}^d)$ on the direct product $\tilde{\mathcal{Q}} := G \times Q$ by

$$A : \begin{array}{ccc} \mathcal{G}^k \times \tilde{\mathcal{Q}} & \longrightarrow & \tilde{\mathcal{Q}} \\ (g, \varphi), (h, q) & \longmapsto & (gh, g \cdot (\varphi \cdot q)) \end{array} \tag{7.13}$$

**Remark 7.13** (Compatibility condition)**.** *Note that the compatibility condition* (7.12) *coincides with condition* (7.5) *when* $\mathcal{Q} = \mathbb{R}^d$. *Therefore condition* (7.12) *is quite natural and is often a direct consequence of the particular condition* (7.5) *when the shape space is one of the classical ones (landmarks, curves, images). Moreover, following remark 7.5, we know that each pair* $(g, \varphi)$ *defines a diffeomorphism* $\varphi_g \circ \varphi \in \mathrm{Diff}_{C^k}(\mathbb{R}^d)$. *Hence condition* (7.12) *often guarantees that the action of* $\mathcal{G}^k$ *can be induced from the deformation action of* $\mathrm{Diff}_{C^k}(\mathbb{R}^d)$.

We prove in the following proposition that the group $\mathcal{G}^k$ and its action on the manifold $\tilde{\mathcal{Q}} = G \times Q$ defines a shape space following the GGA framework 6.1.

**Proposition 7.14** (Shape space $\tilde{Q}$)**.** *The action of the half-Lie group* $\mathcal{G}^k$ *on* $\tilde{\mathcal{Q}} = G \times Q$ *satisfies conditions (S.1-3), i.e.*

1. *The action* $A : \mathcal{G}^k \times \tilde{\mathcal{Q}} \to \tilde{\mathcal{Q}}$ *is continuous.*
2. *For* $(h, q) \in \mathcal{Q}$, *the mapping* $A_{h,q} : (g, \varphi) \mapsto (gh, g \cdot (\varphi \cdot q))$ *is smooth, and we denote* $\xi_{h,q} = T_{(e_G, \mathrm{id})} A_{h,q}$ *its derivative, called the infinitesimal action.*
3. *For* $l > 0$, *the mappings*

$$A : \begin{array}{ccc} \mathcal{G}^{k+l} \times \tilde{\mathcal{Q}} & \longrightarrow & \tilde{\mathcal{Q}} \\ (g, \varphi), (h, q) & \longmapsto & (gh, g \cdot (\varphi \cdot q)) \end{array}$$

*and*

$$\xi : \begin{array}{ccc} T_{(e_G, \mathrm{id})} \mathcal{G}^{k+l} \times \tilde{\mathcal{Q}} & \longrightarrow & T\tilde{\mathcal{Q}} \\ (X, u), (h, q) & \longmapsto & (Xh, X \cdot q + u \cdot q) \end{array}$$

*are* $C^l$.

*Proof.* The action $A : \mathcal{G}^k \times \tilde{\mathcal{Q}} \to \tilde{\mathcal{Q}}$ is continuous since the multiplication is continuous in the Lie group $G$ and the groups $G$ and $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ acts continuously on $Q$.

For $(h, q) \in \mathcal{Q}$, the mapping $A_{(h,q)}$ is smooth since the multiplication is smooth in $G$ and the mappings $g \mapsto g \cdot q$ and $\varphi \mapsto \varphi \cdot q$ are smooth by hypothesis.

For $l > 0$, the mappings $A : \mathcal{G}^{k+l} \times \tilde{\mathcal{Q}} \to \tilde{\mathcal{Q}}$ and $\xi : T_{(e_G, \mathrm{id})} \mathcal{G}^{k+l} \times \tilde{\mathcal{Q}} \to \tilde{\mathcal{Q}}$ are $C^l$. Indeed, the group operation in $G$ is smooth, the action of $G$ on $\mathcal{Q}$ is smooth and the mapping $(\varphi, q) \mapsto \varphi \cdot q$ is $C^l$. □

### 7.2.3 Two variational problems and Hamiltonian dynamic

#### 7.2.3.1 Inexact matching problem

Following the classical LDDMM framework [17], we consider a Hilbert space $V$ of vector fields with continuous inclusion in $C_0^{k+2}(\mathbb{R}^d, \mathbb{R}^d)$. Here, $V$ is not necessarily $G$-invariant. As shown in the previous subsection, we can define a strong sub-Riemannian structure on the group $\mathcal{G}^k$ which allows to perform matching on $\mathcal{Q}$ through the action of $\mathcal{G}^k$. Consider a template shape $q_S \in \mathcal{Q}$ and its corresponding shape $(e_G, q_S) \in \tilde{\mathcal{Q}}$. We recall that given two time-varying vector fields $(X_t, v_t) \in L^2([0,1], \mathfrak{g} \times V)$, we can generate a flow in $(g_t, \varphi_t) \in \mathcal{G}^k$ which defines a trajectory of shape $(h_t, q_t) = (g_t, \varphi_t) \cdot (e_G, q_S)$. The dynamic of the deformed shape under the action of $\mathcal{G}^k$ can be deduced by the infinitesimal action:

$$(\dot{g}_t, \dot{q}_t) = \xi_{g_t, q_t}(X_t, v_t) = (X_t g_t, X_t \cdot q_t + v_t \cdot q_t), \quad (g_0, q_0) = (e_G, q_S) \tag{7.14}$$

The matching of the source shape $(e_G, q_S)$ onto a target $(g_T, q_T)$ can be expressed as an energy minimization problem :

$$\inf_{(X,v)\in L^2([0,1],\mathfrak{g}\times V)} J(X,v) = \int_0^1 \frac{1}{2}|X_t|^2_{\mathfrak{g}} + \frac{1}{2}|v_t|^2_V \, dt + \mathcal{D}(g_1, q_1) \tag{7.15}$$
$$\text{s.t} \quad \begin{cases} \dot{g}_t = X_t g_t \\ \dot{q}_t = X_t \cdot q_t + v_t \cdot q_t \\ (g_0, q_0) = (e_G, q_S) \end{cases}$$

where $\mathcal{D} : \tilde{\mathcal{Q}} \to \mathbb{R}$ is a data attachment term measuring the distance to the target shape. We will discuss in the next section several examples of data attachment terms. We recall the following result on existence of minimizers 6.17 (cf. also appendix 3.3 for a discussion on variational problems on Banach manifold).

**Proposition 7.15** (Existence of minimizers of $J$)**.** *Suppose the data attachment term $\mathcal{D}$ is continuous, then the problem* (7.15) *admits minimizers.*

The end of this section is devoted to the computation and the characterization of the critical points of the energy through an Hamiltonian formulation, similarly to the LDDMM framework. We define the following pre-hamiltonian

$$H(g, q, p^g, p, X, v) = (p^g \mid Xg) + (p \mid v \cdot q + X \cdot q) - \frac{1}{2}|X|^2_{\mathfrak{g}} - \frac{1}{2}|v|^2_V$$

By proposition 6.19, critical points of the energy $J$ satisfies the Hamiltonian equations.

**Proposition 7.16** (Critical points of energy and Hamiltonian dynamics)**.** *Let $(X, v) \in L^2([0,1], \mathfrak{g} \times V)$, and denote $(g_t, q_t)$ the associated curve in $\tilde{\mathcal{Q}}$ satisfying the evolution equation* (7.14)*. Then $(X, v)$ is a critical point of the energy problem* (7.15) *if and only if $(g_t, q_t)$ satisfies the Hamiltonian dynamic*

$$\begin{cases} (\dot{g}_t, \dot{q}_t, \dot{p}^g_t, \dot{p}_t) = \nabla^\omega H(g_t, q_t, p^g_t, p_t, X_t, v_t) \\ \partial_{X,v} H(g_t, q_t, p^g_t, p_t, X_t, v_t)) = 0 \\ (p^g_1, p_1) = -d\mathcal{D}(g_1, q_1) \end{cases}$$

*where $\nabla^\omega H$ is the partial symplectic gradient of $H$ with regards to the canonical weak symplectic form on $T^*\tilde{\mathcal{Q}}$.*

This result is proved in 6.19.

**Remark 7.17.** *In canonical coordinates of the cotangent bundle $T^*\tilde{\mathcal{Q}} = T^*(G \times \mathcal{Q})$, the partial symplectic gradient is given by*

$$\begin{aligned}\nabla^\omega H(g, q, p^g, p, X, v) = \big(&\partial_{p^g} H(g, q, p^g, p, X, v), \partial_p H(g, q, p^g, p, X, v), \\ &- \partial_g H(g, q, p^g, p, X, v), -\partial_q H(g, q, p^g, p, X, v)\big)\end{aligned}$$

By denoting respectively $\xi^{\mathfrak{g}}_q$ and $\xi_q$ the infinitesimal actions of $\mathfrak{g}$ and $V$ on $q$, we define the reduced Hamiltonian by

$$H(g, q, p^g, p) = \frac{1}{2}|K_{\mathfrak{g}}((T_{e_G} R_g)^* p^g + \xi^{\mathfrak{g}*}_q p)|^2_{\mathfrak{g}} + \frac{1}{2}|K_V \xi^{V*}_q p|^2_V \tag{7.16}$$

so that the critical points of the energy $J$ are equivalently expressed as the Hamiltonian flow of $H$

$$(\dot{g}, \dot{q}, \dot{p}^g, \dot{p}) = \nabla^\omega H(g, q, p^g, p)$$

We can notice that the control $X = K_{\mathfrak{g}}((T_{e_G} R_g)^* p^g + \xi_q^{\mathfrak{g}*} p)$ is parameterized by two different covectors which complicates its interpretation. Indeed, in computational anatomy, the covector is a covariable allowing to perform some statistics on the studied deformation. In this framework, we consider two different deformations, so we would like to be able to measure the contribution of each type of deformations separately, which requires to have a separate covariable per deformation. To do so, we introduce a new variable

$$\tilde{q} = g^{-1} \cdot q \tag{7.17}$$

This new variable represents the shape in its intrinsic frame of reference, allowing us to track the non-rigid deformation before it is rigidly transported. Moreover, in the next section, we will focus on reduction which is easier to do considering this change of variable. The idea is that we will remove the influence of $G$ on the shape, in order to only capture the deformation generated by the diffeomorphism. The dynamic of this new shape is given by

$$\dot{\tilde{q}}_t = T_{\mathrm{id}} \rho_{g_t}(v_t) \cdot \tilde{q}_t \tag{7.18}$$

By denoting $\tilde{v}_t = T_{\mathrm{id}} \rho_{g_t}(v_t) \in \tilde{V} = T_{\mathrm{id}} \rho_{g_t}(V)$, the dynamic can be simply expressed through the infinitesimal action $\dot{\tilde{q}}_t = \xi_{\tilde{q}_t}(\tilde{v}_t)$. The dependence on $X$ in the evolution of the shape $\tilde{q}$ is here hidden within the variable $g_t$. Then, the energy associated with the matching problem can be expressed with respect to this new shape.

$$\inf_{(X,v) \in L^2([0,1], \mathfrak{g} \times V)} \tilde{J}(X, v) = \int_0^1 \frac{1}{2} |X_t|^2_{\mathfrak{g}} + \frac{1}{2} |T_{\mathrm{id}} \rho_{g_t^{-1}} \tilde{v}_t|^2_V \, dt + \mathcal{D}(g_1, g_1 \cdot \tilde{q}_1) \tag{7.19}$$

$$\text{s.t} \quad \begin{cases} \dot{g}_t = X_t g_t \\ \dot{\tilde{q}}_t = \xi_{\tilde{q}_t}(\tilde{v}_t) \\ (g_0, \tilde{q}_0) = (e_G, q_S) \end{cases}$$

Similarly to the previous problem, given that the data attachment term is continuous, we can prove existence of minimizers of problem (7.19).

**Proposition 7.18** (Existence of the minimizers of $\tilde{J}$). *Suppose the data attachment term $\mathcal{D}$ is continuous, then the problem* (7.19) *admits minimizers.*

Moreover, the problems (7.15) and (7.19) are equivalent through the change of variable $\tilde{q} = g^{-1} \cdot q$

**Proposition 7.19** (Equivalence of the problems $J$ and $\tilde{J}$). *The minimizers of $J$ are exactly the minimizers of $\tilde{J}$.*

*Proof.* Let $(X^*, v^*) \in L^2(I, \mathfrak{g} \times V)$ be minimizers of $J$. From the dynamic equations associated with $J$ in (7.15), we can show that $\dot{\tilde{q}}_t = T_{\mathrm{id}} \rho_{g_t}(v_t^*) \cdot \tilde{q}_t$. Moreover, since $g_1^{-1} \cdot q_1 = \tilde{q}_1$, we have $J(X^*, v^*) = \tilde{J}(X^*, v^*)$ and thus $\min J \geq \min \tilde{J}$. Similarly, we can prove the reverse inequality, which leads to the equality of the minimizers. $\square$

In a same fashion, the minimizers of the energy $\tilde{J}$ are the geodesics of the Hamiltonian defined by :

$$\tilde{H}(g,\tilde{q},p^g,p',X,v) = (p^g|X\cdot g) + (p|T_{\mathrm{id}}\rho_g(v)\cdot\tilde{q}) - \frac{1}{2}|X|_{\mathfrak{g}}^2 - \frac{1}{2}|v|_V^2$$

The reduced Hamiltonian associated is

$$\tilde{H}(g,\tilde{q},\tilde{p}^g,\tilde{p}) = \frac{1}{2}|K_{\mathfrak{g}}(T_{e_G}R_g)^*\tilde{p}^g|_{\mathfrak{g}}^2 + \frac{1}{2}|K_V(T_{\mathrm{id}}\rho_g)^*\xi_{\tilde{q}}^*\tilde{p}|_V^2$$

We end this section by showing that the two Hamiltonian formulations, $H$ and $\tilde{H}$, are also equivalent

**Proposition 7.20** (Influence of the change of variable on covectors)**.** *Let $A^G$ : $(g,q)\to A(g,q)$ be the action of $G$ on $\mathcal{Q}$, $\xi^{\mathfrak{g}}$ its associated infinitesimal action. Let $(p_0^g,p_0),(\tilde{p}_0^g,\tilde{p}_0)\in T^*_{(e_G,q_S)}\tilde{\mathcal{Q}}$ initial covectors such that $\tilde{p}_0^g = p_0^g + \xi_{q_S}^{\mathfrak{g}*}p_0$ and $\tilde{p}_0 = \partial_2A(g,q_S)^*p$. Let $(g_t,q_t)$ (resp. $(\tilde{g}_t,\tilde{q}_t)$) be the Hamiltonian flow of $H$ (resp. $\tilde{H}$). Then for all time $t$,*

$$\begin{cases} \tilde{g}_t = g_t \\ \tilde{q}_t = g_t\cdot q_t \\ \tilde{p}_t = \partial_2A(g_t,\tilde{q}_t)^*p_t \\ \tilde{p}_t^g = p_t^g + (T_{e_G}R_{g_t^{-1}})^*\xi_{q_t}^{\mathfrak{g}*}p_t \end{cases} \tag{7.20}$$

*Proof.* The proof is given in the next section. □

#### 7.2.3.2 Proof of proposition 7.20

This section is devoted to the proof of proposition 7.20. Let $q_S\in\mathcal{Q}$ be a template shape, and we consider $(e_G,q_S)\in\tilde{\mathcal{Q}} = G\times\mathcal{Q}$ the corresponding shape. We recall the two Hamiltonians on $T^*\tilde{\mathcal{Q}} = T^*G\oplus T^*\mathcal{Q}$.

$$H(g,q,p^g,p) = \frac{1}{2}|K_{\mathfrak{g}}((T_{e_G}R_g)^*p^g + \xi_q^{\mathfrak{g}*}p)|_{\mathfrak{g}}^2 + \frac{1}{2}|K_V\xi_q^*p|_V^2$$

and after the change of variable $\tilde{q} = g^{-1}\cdot q$

$$\tilde{H}(g,\tilde{q},\tilde{p}^g,\tilde{p}) = \frac{1}{2}|K_{\mathfrak{g}}(T_{e_G}R_g)^*\tilde{p}^g|_{\mathfrak{g}}^2 + \frac{1}{2}|K_V(T_{\mathrm{id}}\rho_g)^*\xi_{\tilde{q}}^*\tilde{p}|_V^2$$

We will prove that both these Hamiltonians lead to equivalent dynamic. Let $(p_0^g,p_0),(\tilde{p}_0^g,\tilde{p}_0)\in T^*_{(e_G,q_S)}\tilde{\mathcal{Q}}$ initial covectors such that $\tilde{p}_0^g = p_0^g + \xi_{q_S}^{\mathfrak{g}*}p_0$ and $\tilde{p}_0 = \partial_2A(g,q_S)^*p$. Let $(g_t,q_t)$ (resp. $(\tilde{g}_t,\tilde{q}_t)$) be the Hamiltonian flow of $H$ (resp. $\tilde{H}$). By definition, the Hamiltonian $H$ is induced by the right invariant metric of $\mathcal{G}^k$ and the action of $\mathcal{G}^k$ on the space $\mathcal{Q}$. In particular 6.4, it leads to a momentum map $m: T^*\tilde{\mathcal{Q}}\to(T_e\mathcal{G}^k)^*$ defined by

$$(m(g,q,p^{\mathfrak{g}},p)\mid X,v) = (p^{\mathfrak{g}}\mid T_{e_G}R_g(X)) + (p\mid \xi_q^{\mathfrak{g}}(X) + \xi_q(v))$$

where $(g,q,p^{\mathfrak{g}},p)\in T^*\tilde{\mathcal{Q}}$, and $(X,v)\in\mathfrak{g}\times C_0^k(\mathbb{R}^d,\mathbb{R}^d)$, that defines the momentum trajectory $t\mapsto m_t = m(g_t,q_t,p_t^g,p_t)$ associated with the Hamiltonian flow. The definition of the momentum map here is particularly interesting since it determines the Hamiltonian flow of $H$, and it lives directly in the dual of tangent space at identity of $\mathcal{G}^k$. Its dynamic follows the sub-Riemannian Euler-Poincaré equation 6.20, that we recall in its integrated form:

$$m_t = \mathrm{Ad}^*_{(g_t,\varphi_t)^{-1}}(m_0), \tag{7.21}$$

where $m_0 = (p_0^{\mathfrak{g}}, \xi_{q_S}^{V*} p_0) \in \mathfrak{g}^* \oplus C_0^k(\mathbb{R}^d,\mathbb{R}^d)^*$. In the next step, we prove that the Hamiltonian $\tilde{H}$, after the change of variable, can also be written as the Hamiltonian associated with sub-Riemannian metric on $\tilde{\mathcal{Q}}$ induced by an action of the group $\mathcal{G}^k$. Indeed, we define, for $(g,\varphi) \in \mathcal{G}^k$ and $(g',\tilde{q}) \in \tilde{\mathcal{Q}}$, the action

$$\tilde{A}\left((g,\varphi),(g',\tilde{q})\right) = (gg', \rho_{g'}(\varphi)\cdot\tilde{q})\,.$$

We prove next that this action satisfies conditions of proposition 7.14

**Lemma 7.21** (Second action on $\tilde{\mathcal{Q}}$)**.** *The action $\tilde{A}$ of the half-Lie group $\mathcal{G}^k$ on $\tilde{\mathcal{Q}}$ satisfies the conditions (S.1-3) (6.1).*
*Moreover any curve $(g_t,\tilde{q}_t,p_t^g,p_t)$ that is the Hamiltonian flow of $\tilde{H}$ in $T^*\tilde{\mathcal{Q}}$, with initial condition $(g_0,\tilde{q}_S) = (e_G,q_S)$, can be lifted to a curve $(g'_t,\varphi_t,p_t'^g,p_t^\varphi)$ in the space $T^*\mathcal{G}^k$ such that*

1. $(p_0'^g, p_0^\varphi) = (p_0^g, \xi_{\tilde{q}}^{V*} p_0^\varphi)$
2. $(g_t,\tilde{q}_t) = (g'_t,\varphi_t)\cdot(e_G,q_S)$*, and in particular* $g_t = g'_t$.

*Proof.* The first part of the proof is similar to the proof of proposition 7.14, since the regularity conditions of the action $\tilde{A}$ follows from the structure of the semidirect product $\mathcal{G}^k$ and on the regularity asumptions on all the different actions. Moreover, as we saw, the Hamiltonian $\tilde{H}$ comes from the pre-Hamiltonian

$$\tilde{H}(g',\tilde{q},p^g,p,X,v) = \Big(p^g, p \mid \tilde{\xi}_{g',\tilde{q}}(X,v)\Big) - \frac{1}{2}(|X|^2_{\mathfrak{g}} + |v|^2_V)$$

so that the Hamiltonian $\tilde{H}$ is induced by the right invariant metric on $\mathcal{G}^k$ and the action of the group $\mathcal{G}^k$ on $\tilde{Q}$. The last part of the lemma follows then from theorem 6.21. $\square$

In particular, this action also induces a momentum map $\tilde{m} : T^*\tilde{\mathcal{Q}} \to T_e^*\mathcal{G}^k$ defined by

$$\tilde{m}(g,\tilde{q},p^g,\tilde{p})(X,v) = (p^g|T_eR_gX) + (\tilde{p}|\xi_{\tilde{q}}T_{\mathrm{id}}\rho_g v),$$

and if $(g_t,\tilde{q}_t,p_t^g,\tilde{p}_t)$ is the Hamiltonian flow of $\tilde{H}$, then the momentum $m_t = \tilde{m}(g_t,\tilde{q}_t,p_t^g,\tilde{p}_t)$ also satisfies the sub-Riemannian Euler-Poincaré-Arnold equation (5.11), and this totally determines the flow $(g_t,\tilde{q}_t,p_t^g,\tilde{p}_t)$. In particular if the initial momenta $m_0$ and $\tilde{m}_0$ associated with a flow of $H$ and $\tilde{H}$ are equal, then the momenta $m_t$ and $\tilde{m}_t$ are equal for all time and they lead to the same curve in $T^*\mathcal{G}^k$. We get therefore the following result.

**Lemma 7.22.** *Let $(p_0^g,p_0),(\tilde{p}_0^g,\tilde{p}_0) \in T^*_{(e_G,q_S)}\tilde{Q}$ initial covectors such that $\tilde{p}_0^g = p_0^g + \xi_{q_S}^{\mathfrak{g}*}p_0$ and $\tilde{p}_0 = \partial_2 A(g,q_S)^*p_0$. Let $(g_t,q_t)$ (resp. $(\tilde{g}_t,\tilde{q}_t)$) be the Hamiltonian flow of $H$ (resp. $\tilde{H}$). Then for all time $t$,*

$$\begin{cases} g_t = \tilde{g}_t \\ q_t = g_t\cdot\tilde{q}_t \\ (T_{e_G}R_{g_t})^*p_t^g + \xi_{q_t}^{\mathfrak{g}*}p_t = (T_{e_G}R_{g_t})^*\tilde{p}_t^g \\ \xi_{q_t}^{V*}p_t = (T_{\mathrm{id}}\rho_{g_t})^*\xi_{\tilde{q}_t}^{V*}\tilde{p}_t \end{cases} \tag{7.22}$$

*Proof.* We use the momenta trajectories to prove this result. We define the momentum $m_t = m(g_t, q_t, p_t^g, p_t)$ associated with the flow of $H$, and $\tilde{m}_t = \tilde{m}(\tilde{g}_t, \tilde{q}_t, \tilde{p}_t^g, \tilde{p}_t)$ associated with $\tilde{H}$. In particular, we have $m_0 = (p_0^g + \xi_{q_S}^{\mathfrak{g}*} p_0, \xi_{q_S}^{V*} p_0)$, and $\tilde{m}_0 = (p_0^g, \xi_{q_S}^{V*} \tilde{p}_0)$, so that, by asumptions

$$m_0 = \tilde{m}_0$$

But then, by theorem 6.23, we have for all $t$, $m_t = \tilde{m}_t$ which proves the lemma. $\square$

Note that this lemma is not sufficient to prove proposition 7.20. Indeed, we can notice that $T_{\mathrm{id}}\rho_{g_t}\xi_{\tilde{q}_t}^{V^*} = \xi_{q_t}^{V^*}\partial_2 A(g_t^{-1}, q_t)^*$ thanks to the compatibility condition. Consequently the second equation can be expressed by :

$$\xi_{q_t}^{V*} p_t = \xi_{q_t}^{V*} \partial_2 A(g_t^{-1}, q_t)^* \tilde{p}_t$$

However, we cannot deduce directly an equality between the moments since the operator $\xi_q$ is not necessarily invertible. Thus, we will prove this relation and proposition 7.20 by direct calculus:

*Proof.* By lemma 7.22, we directly get

$$\begin{cases} \tilde{g}_t = g_t \\ \tilde{q}_t = g_t \cdot q_t \\ \tilde{p}_t^g = p_t^g + (T_{e_G} R_{g_t^{-1}})^* \xi_{q_t}^{\mathfrak{g}*} p_t \\ \xi_{q_t}^{V*} p_t = \xi_{q_t}^{V*} \partial_2 A(g_t^{-1}, q_t)^* \tilde{p}_t \end{cases}$$

Since $T_e R_g$ is invertible, we get the equality

$$\tilde{p}_t^g = p_t^g + (T_{e_G} R_{g_t^{-1}})^* \xi_{q_t}^{\mathfrak{g}*} p_t.$$

It remains to prove the equality on the covectors $p_t$ and $\tilde{p}_t$. Let $A_t = \partial_2 A(g_t^{-1}, q_t)$, be an invertible operator with inverse $A_t^{-1} = \partial_2 A(g_t, \tilde{q}_t)$. In particular, we can relate this operator to the infinitesimal action.

$$\xi_{\tilde{q}_t}^V(\tilde{v}_t) = A_t \xi_{q_t}^V(v_t) \quad \text{or equivalently} \quad \xi_{q_t}^V(v_t) = A_t^{-1} \xi_{\tilde{q}_t}^V(\tilde{v}_t) \tag{7.23}$$

We will prove the first equality of 7.20 by showing that $\tilde{p}_t$ and $A_t^{-1*} p_t$ follow the same dynamic. For $\delta q \in T_{\tilde{q}} Q$

$$\partial_t (A_t^{-1*} p_t | \delta q) = -(p_t | (\partial_q \xi_{q_t}(v_t) + \partial_q \xi_{q_t}^{\mathfrak{g}}(X_t)) A_t^{-1} \delta q) + (p_t | \partial_t A_t^{-1} \delta q) \tag{7.24}$$

where the time derivative of $A_t^{-1}$ is

$$\partial_t A_t^{-1} = \partial_{1,2}^2 A(g_t, \tilde{q}_t) T_e R_{g_t}(X_t) + \partial_{2,2}^2 A(g_t, \tilde{q}_t) \dot{\tilde{q}}_t$$

By noticing that $\partial_1 A(g_t, \tilde{q}_t) T_e R_{g_t}(X_t) = \xi_{g_t q_t}^{\mathfrak{g}}(X_t)$, it follows that

$$\partial_{1,2}^2 A(g_t, \tilde{q}_t) T_e R_{g_t}(X_t) = \partial_q \xi_{\tilde{q}_t}^{\mathfrak{g}}(X_t) A_t^{-1}$$

such that we can simplify the previous expression :

$$\partial_t A_t^{-1} = \partial_q \xi_{\tilde{q}_t}^{\mathfrak{g}}(X_t) A_t^{-1} + \partial_{2,2}^2 A(g_t, \tilde{q}_t) \dot{\tilde{q}}_t$$

Then, by substitution in Eq.7.24,

$$\partial_t(p_t|A_t^{-1}\delta q) = (p_t|\partial_{2,2}^2 A(g_t,\tilde{q}_t)(\dot{\tilde{q}}_t,\delta q) - \partial_q\xi_{q_t}(v_t)A_t^{-1}\delta q)$$

and using the relations from (7.23), we can deduce that

$$\partial_q\xi_{q_t}(v_t)A_t^{-1} = \partial_{2,2}^2 A(g_t,\tilde{q}_t)\dot{\tilde{q}}_t + A_t^{-1}\partial_q(\xi_{\tilde{q}_t}(\tilde{v}_t)).$$

We can conclude that

$$\partial_t(A_t^{-1*}p_t|\delta q) = -(\partial_q(\xi_{\tilde{q}_t}(\tilde{v}_t))^* A_t^{-1*}p_t|\delta q),$$

which leads to

$$\frac{d}{dt}(A_t^{-1*}p_t) = -\partial_q(\xi_{\tilde{q}_t}(\tilde{v}_t))^*(A_t^{-1*}p_t).$$

Finally, the result follows by the uniquess of the solution under equality of initial conditions thanks to Picard-Lindelöf theorem. □

**Remark 7.23.** *Even though, the relation between $p$ and $\tilde{p}$ corresponds to the lift of the action $A_{g^{-1}}(q) := A(g^{-1},q)$ on the cotangent bundle $T^*Q$, the mapping*

$$\Phi : \begin{array}{ccc} T^*(G\times Q) & \longrightarrow & T^*(G\times Q) \\ (g,q,p^{\mathfrak{g}},p) & \longmapsto & (g, g^{-1}\cdot q, p^{\mathfrak{g}} + (T_{e_g}R_{g^{-1}})^*\xi_q^{\mathfrak{g}*}p, \partial_2 A(g,g^{-1}\cdot q)^*p) \end{array}$$

*is not a symplectomorphism.*

#### 7.2.3.3 Choice of the data attachment term

In a registration task, the purpose is to match a template shape $q_S$ onto a target shape $q_T$. However, the data attachment term depends on both the finite-dimensional group and the shape space $\mathcal{Q}$. By proposition 7.16, covectors are related to the data attachment term by the relation

$$(p_1^g, p_1) = -d\mathcal{D}(g_1,q_1),$$

so the choice of the data attachment term will have consequences on the expression of the geodesics.

- A natural choice for the data attachment term is to consider one that depends only on the shape $\mathcal{D}(g,q) = \mathcal{D}(q)$ since it is the object we want to match. It follows that $p_1^g = 0$ which implies that $p_t^g = 0$ for every time $t \in [0,1]$ (or equivalently, after change of variable (7.17), we get for all $t$, that $\tilde{p}_t^g$ is a variable depending on $(g_t,\tilde{p}_t,\tilde{q}_t)$). This means that we can get rid of the covector $p_t^g$, and that the dynamic of $g_t$ and of $q_t$ is only given by $p_t$ : we do not obtain in that case a decoupling of the deformations.

- Another possible choice is to split the data attachment term in two separate data attachment terms, one for the group element and one for the shape $\mathcal{D}(g,q) = \mathcal{D}^G(g) + \mathcal{D}^Q(q)$. Then, $(p_1^g,p_1) = -(d\mathcal{D}^G(g_1), d\mathcal{D}^G(q_1))$ and the covectors are not related.

- Considering the shape $\tilde{q}$ in its frame of reference (after change of variable (7.17)), the previous data attachment term then mixes the group element and the shape and becomes : $\mathcal{D}(g,\tilde{q}) = \mathcal{D}^G(g) + \mathcal{D}^Q(g\cdot\tilde{q})$.

- The data attachment can also be invariant under the action of G, i.e $\mathcal{D}(g, g\cdot\tilde{q}) = \mathcal{D}(g,\tilde{q}) = \mathcal{D}^G(g) + \mathcal{D}^Q(\tilde{q})$

### 7.2.4 Reduction of the dynamic

In this section, we use symmetries in the Hamiltonian in order to decouple the action of $G$ and of the group of diffeomorphisms on shapes. In many case, the deformation induced by the finite dimensional Lie group $G$ can be performed by diffeomorphisms, and therefore we want to avoid the learning by diffeomorphisms of the motion that could be performed only by $G$. We suppose in all this section that the space $V$ is $G$-invariant, meaning that for all $g \in G$, the mapping $T_{\mathrm{id}}\rho_g$ preserves $V$ and is an isometry. This induces symmetries in the Hamiltonian and allows to perform symplectic reduction as in [66, 65, 85],

Let recall how the symplectic reduction can be performed in our setting and some properties of the Marsden-Weinstein symplectic quotient. The action of the Lie group G on $\mathcal{Q}$ can be lifted by symplectomorphisms on $T^*\mathcal{Q}$ with action given by :

$$g \cdot (q, p) = (g \cdot q, g^{-1} \cdot^* p).$$

meaning that for any $h \in T_{g \cdot q}\mathcal{Q}$, we have

$$\big(g^{-1} \cdot^* p \mid h\big) = \big(p \mid g^{-1} \cdot h\big)$$

This action is actually an Hamiltonian action (cf. 3.1.3.1). Indeed, by denoting $\Gamma(T^*\mathcal{Q})$ the space of smooth sections of the cotangent bundle $T^*\mathcal{Q}$, we introduce the infinitesimal action

$$\xi^G : \begin{array}{ccc} \mathfrak{g} & \longrightarrow & \Gamma(T^*\mathcal{Q}) \\ X & \longmapsto & \Big((q,p) \mapsto \frac{\partial}{\partial g}\big(g \cdot (q,p)\big)_{|g=e} X\Big) \end{array},$$

and define the mapping $\hat{\mu} : \mathfrak{g} \to C^\infty(T^*\mathcal{Q}, \mathbb{R})$ by

$$\hat{\mu}(X)(q, p) = (p \mid X \cdot q).$$

The mapping $\hat{\mu}$ defines for any $X \in \mathfrak{g}$, a Hamiltonian function $\hat{\mu}(X) : T^*\mathcal{Q} \to \mathbb{R}$ on $T^*\mathcal{Q}$, and the infinitesimal action $\xi^G(X) \in \Gamma(T^*\mathcal{Q})$ gives its Hamiltonian vector field, i.e.

$$\nabla^\omega \hat{\mu}(X) = \xi^G(X).$$

The mapping $\hat{\mu}$ gives also rise to a momentum map $\mu : T^*\mathcal{Q} \to \mathfrak{g}^*$ defined by $(\mu(q,p) \mid X) = \hat{\mu}(X)(q,p)$. Moreover, this momentum map is also $G$-equivariant since we have

$$\begin{aligned} (\mu(g \cdot (q,p)) \mid X) &= \big(p \mid g^{-1} \cdot (X \cdot (g \cdot q))\big) \\ &= (p \mid \mathrm{Ad}_{g^{-1}}(X) \cdot q) \\ &= (\mathrm{Ad}^*_{g^{-1}}\, \mu(q,p) \mid X) \end{aligned}$$

Consequently, the action of $G$ on $T^*Q$ is an Hamiltonian action (cf. [65] for more details on Hamiltonian action). In addition, if we assume that the action of $G$ on $\mathcal{Q}$ is proper and free, we can consider the Marsden–Weinstein quotient $\mu^{-1}(0)/G$ (also sometimes denoted by $T^*\mathcal{Q}//G$) which is a symplectic manifold equipped with the induced weak symplectic form. The notation $T^*\mathcal{Q}//G$ means that to quotient out by the action of $G$, we must follow a 2-step reduction process : we first restrict from $T^*\mathcal{Q}$ to $\mu^{-1}(0)$ and we reduce again via the quotient $\mu^{-1}(0)/G$. Note that in this particular case, we get an identification [85]

$$\mu^{-1}(0)/G \simeq T^*(Q/G).$$

Moreover, if $V$ is $G$-invariant, then the Hamiltonian $H$ is also $G$-invariant, and therefore, by Noether's theorem 3.25, the momentum is conserved on the flow of $H$.

We now adapt this framework to the shape space $\tilde{Q} = G \times \mathcal{Q}$. In this setting, the group $G$ acts only on the second component of $G \times \mathcal{Q}$ :

$$g \cdot (g', q) = (g', g \cdot q)$$

and we perform in the same fashion a symplectic reduction with regards to this action. This action of $G$ on $G \times \mathcal{Q}$ can be lifted by symplectomorphism on $T^*G \oplus T^*\mathcal{Q}$, and will lead to the same momentum map and reduction process. Under the same hypothesis (free and proper action of $G$ on $\mathcal{Q}$), we can deduce that the space

$$T^*G \oplus \mu^{-1}(0)/G$$

defines a symplectic manifold. As stated in the following proposition, a first consequence is that, since the Hamiltonian $H : T^*G \oplus T^*Q \to \mathbb{R}$ defined in (7.16) is invariant by the action of $G$, it can be reduced to an Hamiltonian on the reduced space $T^*G \oplus \mu^{-1}(0)/G$.

**Proposition 7.24** (Projection of the hamiltonian flow [66])**.** *Let $i : T^*G \oplus \mu^{-1}(0) \hookrightarrow T^*G \oplus T^*\mathcal{Q}$ be the inclusion map and $\pi : T^*G \oplus \mu^{-1}(0) \to T^*G \oplus \mu^{-1}(0)/G$ denote the canonical projection. Let $H : T^*G \oplus T^*\mathcal{Q} \to \mathbb{R}$ be a Hamiltonian which invariant under the action of $G$. Then the hamiltonian flow on $T^*G \oplus \mu^{-1}(0)$ associated with the hamiltonian $H$ induces an hamiltonian flow on $T^*G \oplus \mu^{-1}(0)/G$ whose hamiltonian $\bar{H}$ is defined by $\bar{H} \circ \pi = H \circ i$.*

Furthermore, since we get the isomorphism

$$T^*G \oplus \mu^{-1}(0)/G \simeq T^*G \oplus T^*(\mathcal{Q}/G),$$

the Hamiltonian flow of $H$ on $T^*G \oplus \mu^{-1}(0)$ can be projected on an Hamiltonian flow on $T^*G \oplus T^*(\mathcal{Q}/G)$ which can be interpreted as a decoupling between the diffeomorphism part and the action of $G$.

**Variational problem and reduction.** We finish this part by linking this reduction process with the variational problems we are interested in. First, since $V$ is $G$-invariant, for any $v \in V$ we have that $|v|_V = |d_{\mathrm{id}}\rho_g v|_V^2$ for all $g \in G$. Recall the functional (7.19) that we wants to minimize is

$$J(X, v) = \int_I |X_t|_{\mathfrak{g}}^2 + |v_t|_V^2 \, dt + \mathcal{D}(g_1, \tilde{q}_1).$$

We first start by supposing the the data attachment term $\mathcal{D} : G \times \mathcal{Q} \to \mathbb{R}$ is $G$-invariant, i.e. $\forall g' \in G,\ \mathcal{D}(g, g' \cdot q) = \mathcal{D}(g, q)$. In particular the mapping $\mathcal{D}$ induces a reduced data attachment term on $\bar{\mathcal{Q}} = G \times \mathcal{Q}/G$ with associated equivalence class $[q]$, that we denote $\bar{\mathcal{D}}$. In this case minimizing $J$ becomes equivalent to minimizing

$$\bar{J}(X, v) = \int_I |X_t|_{\mathfrak{g}}^2 + |v_t|_V^2 \, dt + \bar{\mathcal{D}}(g_1, [\tilde{q}_1]),$$

that is to say to perform the registration directly in the space $G \times \mathcal{Q}/G$, leading to the reduced hamiltonian flow in $T^*G \oplus T^*(\mathcal{Q}/G)$.

However, the asumption of a data attachment term $\mathcal{D}$ being $G$-invariant is a bit strong in practice. A first idea would be to replace the term $\mathcal{D}$ by the $G$-invariant term

$$\bar{\mathcal{D}}(\tilde{q}) = \inf_{g\in G} \mathcal{D}(g_1, g\cdot\tilde{q})$$

This term could be computationally demanding, and we might want to keep $\mathcal{D}$ in the optimization. Nonetheless, by [9], minimizing $J$ is equivalent to minimizing

$$\tilde{J}_2(p_0^g, p_0) = \tilde{H}(e_G, q_S, p_0^g, p_0) + \mathcal{D}\left(g_1, \tilde{q}_1\right)$$

where we shoot from $(e_G, q_S)$ by the initial covectors $(p_0^g, p_0)$. Moreover, since $V$ is $G$-invariant, we recall that the Hamiltonian can be rewritten as a sum of Hamiltonian functions on $T^*G$ and $T^*\mathcal{Q}$

$$\tilde{H}(g, \tilde{q}, p^g, p) = H^G(g, p^g) + H^V(\tilde{q}, p).$$

where $H^G(g, p^g) = \frac{1}{2}|K_V \xi_{\tilde{q}}^* p|_V^2$ and $H^V(\tilde{q}, p) = \frac{1}{2}|K_{\mathfrak{g}} T_e R_g^* p^g|_{\mathfrak{g}}^2$. Therefore, in this setting, we can still restrict to initial covectors $p_0$ in $\mu^{-1}(0)$ in order to get hamiltonian flow that descends to the reduced space.

### 7.2.5 Application to rigid and diffeomorphic motions

In this section, we focus on the example of the rigid motions represented by the group of isometries and its action on space of curves.

#### 7.2.5.1 The semidirect product of isometries and diffeomorphisms

The group of isometries of $\mathbb{R}^d$ is a finite-dimensional Lie group particularly interesting in the computational anatomy framework since they represent classic and simple deformations. We denote $\mathrm{Isom}(\mathbb{R}^d) = SO_d \ltimes \mathbb{R}^d$ the group of isometries and its associated Lie algebra $\mathfrak{isom}_d = \mathrm{Skew}_d \oplus \mathbb{R}^d$. First, we can notice that the group of isometries acts on $\mathbb{R}^d$ smoothly via diffeomorphisms and properly by $(R, T)\cdot x = Rx + T$. Moreover, the action of the group $\mathrm{Isom}(\mathbb{R}^d)$ on $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ defined by

$$\rho_{(R,T)}(\varphi)(x) = R^\top \varphi(Rx + T) - R^\top T,\ x \in \mathbb{R}^d \tag{7.25}$$

satisfies the assumptions of the framework presented in the subsection 7.2.1.1. Then, we can consider the semidirect product $\mathrm{Isom}(\mathbb{R}^d) \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$, which is a Banach half-Lie group, with group operations

$$\begin{cases} (R', T', \varphi')(R, T, \varphi) = (R'R, R'T + T', R^\top \varphi'(R\varphi + T) - R^\top T) \\ (R, T, \varphi)^{-1} = (R^\top, -R^\top T, R\varphi^{-1}(R^{-1} - R^\top T) + T) \end{cases} \tag{7.26}$$

Given $(A, \tau, u) \in L^2([0,1], \mathfrak{isom}_d \oplus C_0^k(\mathbb{R}^d, \mathbb{R}^d))$, a flow in $\mathrm{Isom}(\mathbb{R}^d) \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ is defined by :

$$\begin{cases} \dot{R}_t = A_t R_t \\ \dot{T}_t = \tau_t + A_t T_t \\ \dot{\varphi}_t = R_T^\top u_t(R_t\varphi_t + T_t) \\ (R_0, T_0, \varphi_0) = (I_d, 0, \mathrm{id}) \end{cases}$$

#### 7.2.5.2 The shape space of curves

For $k \geq 1$, let $D$ be either the interval $[0,1]$ for open curves or the unit circle $\mathbb{S}^1$ for closed curves. The space of $C^k$-immersions in $\mathbb{R}^d$,

$$\mathrm{Imm}_{C^k}(D, \mathbb{R}^d) = \{q \in C^k(D, \mathbb{R}^d) \mid q'(t) \neq 0 \text{ for all } t \in D\}$$

is an open set of the Banach space $C^k(D, \mathbb{R}^d)$ which induces a manifold structure with tangent space $T_q \mathrm{Imm}_{C^k}(D, \mathbb{R}^d) \simeq C^k(D, \mathbb{R}^d)$, the space of $C^k$ vector fields along $q$. Reparameterization of curves arises from the action of the group of orientation-preserving diffeomorphism $\mathrm{Diff}^+(D)$ on the space of immersions :

$$\begin{array}{ccc} \mathrm{Diff}^+(D) \times \mathrm{Imm}_{C^k}(D, \mathbb{R}^d) & \longrightarrow & \mathrm{Imm}_{C^k}(D, \mathbb{R}^d) \\ (\varphi, q) & \longmapsto & q \circ \varphi \end{array}$$

The space of curves that we would originally consider is the space of immersions modulo reparameterizations $\mathcal{S}(D, \mathbb{R}^d) = \mathrm{Imm}_{C^k}(D, \mathbb{R}^d) / \mathrm{Diff}^+(D)$, however since we will use a varifold data attachment term, which is invariant by reparameterization, we can identify the space of curves as the space of $C^k$-immersions . The group of diffeomorphism $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ acts on $\mathrm{Imm}_{C^k}(D, \mathbb{R}^d)$ by left composition $(\varphi, q) \mapsto \varphi \circ q$ with infinitesimal action $\xi : (X, q) \mapsto X \circ q$ which makes $\mathrm{Imm}_{C^k}(D, \mathbb{R}^d)$ a shape space in the sense of [6]. Moreover, the group of isometries acts on curves via

$$(R, T) \cdot q = Rq + T$$

which allows to define a shape space structure on curves with respect to the semidirect product of isometries by diffeomorphisms as stated in the following proposition.

**Proposition 7.25** (Shape space $\mathrm{Isom}(\mathbb{R}^d) \times \mathrm{Imm}_{C^k}(D, \mathbb{R}^d)$)**.** *The action of* $\mathrm{Isom}(\mathbb{R}^d)$ *on* $\mathrm{Imm}_{C^k}(D, \mathbb{R}^d)$ *is a smooth action via diffemorphisms and satisfies the compatibility condition*

$$\varphi \cdot ((R, T) \cdot q) = (R, T) \cdot (\rho_{(R,T)}(\varphi) \cdot q)$$

*Consequently,* $\mathrm{Isom}(\mathbb{R}^d) \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ *induces a shape space structure on* $\mathrm{Isom}(\mathbb{R}^d) \times \mathrm{Imm}_{C^k}(D, \mathbb{R}^d)$.

*Proof.* The smoothness and the inversibility of the action can be deduced directly from its expression. Using the definitions of the differents actions, the left and right hand side of the compatibility condition are respectively equal. Indeed, on one hand we have

$$\varphi \cdot ((R, T) \cdot q) = \varphi(Rq + T)$$

and on the other hand,

$$\begin{aligned} (R, T) \cdot (\rho_{(R,T)}(\varphi) \cdot q) &= (R, T) \cdot (R^\top \varphi(Rq + T) - R^\top T) \\ &= R(R^\top \varphi(Rq + T) - R^\top T) + T \\ &= \varphi(Rq + T). \end{aligned}$$

Finally, it follows from proposition 7.14 that $\mathrm{Isom}(\mathbb{R}^d) \times \mathrm{Imm}_{C^k}(D, \mathbb{R}^d)$ is a shape space. $\square$

In particular, the action of $\mathrm{Isom}(\mathbb{R}^d) \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ on $\mathrm{Isom}(\mathbb{R}^d) \times \mathrm{Imm}_{C^k}(D, \mathbb{R}^d)$ is defined by

$$(R, T, \varphi) \cdot (R', T', q) = (RR', RT' + T, R(\varphi \circ q) + T)$$

and leads to the following infinitesimal action, for $(A, \tau, v) \in \mathfrak{isom}_d \oplus C_0^k(\mathbb{R}^d, \mathbb{R}^d)$:

$$(A, \tau, v) \cdot (R, T, q) = (AR, AT + \tau, Aq + \tau + v \circ q)$$

#### 7.2.5.3 Matching problem and reduction

Let $V \hookrightarrow C_0^{k+2}(\mathbb{R}^d, \mathbb{R}^d)$ be a RKHS induced by the gaussian kernel $k_\sigma(x, y) = \exp(-\frac{\|y-x\|^2}{2\sigma^2})$, i.e the Hilbert space defined by the completion of the vector space spanned by the functions $k_\sigma(x, \cdot) : y \mapsto k_\sigma(x, y)$ [39]. Let also $q_S \in \mathrm{Imm}_{C^k}(D, \mathbb{R}^d)$ a source curve that we want to match to a target curve $q_T$. We consider the following minimization problem

$$\inf_{(A,\tau,v)} J(A, \tau, v) = \int_0^1 \frac{1}{2}|A_t|^2 + \frac{1}{2}|\tau_t|^2 + \frac{1}{2}|v_t|^2 \, dt + \mathcal{D}((R_1, T_1), R_1\tilde{q}_1 + T_1) \quad (7.27)$$

$$\text{s.t} \quad \begin{cases} \dot{R}_t = A_t R_t \\ \dot{T}_t = A_t T_t + \tau_t \\ \dot{\tilde{q}}_t = R_t^{-1} v_t(R_t \tilde{q}_t + T_t) \\ (R_0, T_0, \tilde{q}_0) = (I_d, 0, q_S) \end{cases}$$

where $\tilde{q} = R^{-1}(q - T)$. This allows to perform a registration between the source curve $q_S$ and the target, taking into account both rigid and non rigid deformations.

**Proposition 7.26** (Hamiltonian associated with (7.27))**.** *The hamiltonian associated with the variational problem* (7.27) *is*

$$H(R, T, \tilde{q}, \tilde{p}^A, \tilde{p}^\tau, \tilde{p}) = \frac{1}{2}(|\tilde{p}^A R^\top + \tilde{p}^\tau T^\top|^2 + |\tilde{p}^\tau|^2 + |K_V \xi_{\tilde{q}}^* \tilde{p}|^2) \quad (7.28)$$

*and the critical points of the energy satisfy the geodesic equations.*

*Proof.* The pre-Hamiltonian associated with this problem is

$$H(R, T, \tilde{q}, \tilde{p}^A, \tilde{p}^\tau, \tilde{p}, A, \tau, v) = (\tilde{p}^A | AR) + (\tilde{p}^\tau | AT + \tau) + (\tilde{p} | R^{-1} v(R\tilde{q} + T)) - \frac{1}{2}(|A|^2 + |\tau|^2 + |v|^2)$$

The geodesic equations lead to the following Hamiltonian

$$H(R, T, \tilde{q}, \tilde{p}^A, \tilde{p}^\tau, \tilde{p}) = \frac{1}{2}|\tilde{p}^A R^\top + \tilde{p}^\tau T^\top|^2 + \frac{1}{2}|\tilde{p}^\tau|^2 + |K_V \xi_{R\tilde{q}+T}^*(R\tilde{p})|^2$$

The invariance by isometries of the kernel implies that $|K_V \xi_{R\tilde{q}+T}^*(R\tilde{p})| = |K_V \xi_{\tilde{q}}^* \tilde{p}|$. Therefore, the critical points of the energy satisfy the geodesic equations following Prop. 7.16. $\square$

Thanks to this invariance, we can perform reduction as presented in the previous section. The action of isometries on immersions $(R, T) \cdot q = Rq + T$ is proper but is not necessarily free. Indeed, certain symmetric immersions, such as straight lines or circles, are stabilized by nontrivial subgroups of $\mathrm{SO}(d)$. To overcome this issue, we consider the space of immersions that have trivial stabilizer.

$$\mathrm{Imm}_{C^k}(D, \mathbb{R}^d)_e = \{q \in \mathrm{Imm}_{C^k}(D, \mathbb{R}^d) \mid \mathrm{Stab}(q) = \{e\}\} \quad (7.29)$$

This assumption is not too restrictive since this space is an open set of the space of immersions, so it is a manifold too, and is dense in $\mathrm{Imm}_{C^k}(D,\mathbb{R}^d)$, so it still represents almost all the curves we will be interested in. Moreover, the invariance of the kernel implies that the space $V$ is $\mathrm{Isom}(\mathbb{R}^d)$-invariant under the action of $\mathrm{Isom}(\mathbb{R}^d)$, i.e $d_{\mathrm{id}}\rho_{(R,T)} : v \mapsto (x \mapsto R^{-1}v(Rx+T))$ is an isometry and preserves $V$. Then, we define the momentum map $\mu : T^* \mathrm{Imm}(\mathrm{D},\mathbb{R}^{\mathrm{d}})_e \to \mathfrak{isom}_d^*$ by

$$\begin{aligned}(\mu(q,p)|(A,\tau)) &= (\xi_q^{\mathfrak{g}*}p|(A,\tau)) \\ &= \frac{1}{2}\int_D \mathrm{tr}((q(s)p(s)^\top - p(s)q(s)^\top)A)ds + \int_D p(s)^\top \tau \, ds \\ &= (\mu_A(q,p)|A) + (\mu_\tau(q,p)|\tau)\end{aligned}$$

The momentum map $\mu_A$ corresponds to the angular momenta in classical mechanic, denoted $p \wedge q$. Moreover, Noether's theorem states that the momentum maps $\mu_A(q,p)$ and $\mu_\tau(q,p)$ are constants of the motion, which we can retrieve by direct computations.

$$\begin{aligned}\frac{d}{dt}\mu_\tau(\tilde{q}_t,\tilde{p}_t) &= \int_D \dot{\tilde{p}}_t(s)ds \\ &= -\int_D\int_D \nabla k_\sigma(\tilde{q}(s),\tilde{q}(s'))\langle\tilde{p}(s),\tilde{p}(s')\rangle + \int_D A\tilde{p}(s)^\top \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\mu_A(\tilde{q}_t,\tilde{p}_t) &= \int_D \dot{\tilde{q}}(s)\tilde{p}(s)^\top + \tilde{q}(s)\dot{\tilde{p}}(s)^\top - (\dot{\tilde{p}}(s)\tilde{q}(s)^\top + \tilde{p}(s)\dot{\tilde{q}}(s)^\top))ds \\ &= \int_D\int_D k(\tilde{q}(s),\tilde{q}(s'))\tilde{p}(s')\tilde{p}(s)^\top - \tilde{q}(s)\nabla k(\tilde{q}(s),\tilde{q}(s'))^\top\langle\tilde{p}(s),\tilde{p}(s')\rangle dsds' \\ &+ \int_D\int_D \nabla k(\tilde{q}(s),\tilde{q}(s'))\tilde{q}(s)^\top\langle\tilde{p}(s),\tilde{p}(s')\rangle - k(\tilde{q}(s),\tilde{q}(s'))\tilde{p}(s)\tilde{p}(s')^\top dsds' \\ &= 0\end{aligned}$$

So, if we enforce that $(\mu_A,\mu_\tau) = 0$ at $t = 0$, then $(\mu_A,\mu_\tau) = 0$ for all time t. Thus, by restricting to the zero level set of the momentum map, we can define a reduced Hamiltonian on

$$\mathfrak{isom}_d^* \times ((\mu_A,\mu_\tau)^{-1}(0)/\mathrm{Isom}(\mathbb{R}^d)) \simeq \mathfrak{isom}_d^* \times (T^*\mathrm{Imm}_e(D,\mathbb{R}^d)/\mathrm{Isom}(\mathbb{R}^d)) \tag{7.30}$$

thanks to the canonical projection

$$\pi : \mathfrak{isom}_d^* \times (\mu_A,\mu_\tau)^{-1}(0) \to \mathfrak{isom}_d^* \times (\mu_A,\mu_\tau)^{-1}(0)/\mathrm{Isom}(\mathbb{R}^d)$$

Coming back to an inexact match problem, we want to minimize the functional on $(\mu_A,\mu_\tau)^{-1}(0)$ using a two-term data attachment:

$$J(p_0^A,p_0^\tau,p_0) = H(p_0^A,p_0^\tau,p_0) + \mathcal{D}_{\mathrm{rigid}}(R_1,T_1) + \mathcal{D}(R_1\tilde{q}_1 + T_1)$$

**Remark 7.27.** *Since $(p_0^A,p_0^\tau,p_0) \in (\mu_A,\mu_\tau)^{-1}(0)$, then we simply have*

$$H(p_0^A,p_0^\tau,p_0) = \frac{1}{2}\left(|p_0^A|^2 + |p_0^\tau|^2 + |K_V\xi_{q_0}^*p_0|^2\right)$$

#### 7.2.5.4 Numerical applications

For numerical applications, curves in $\mathbb{R}^d$ are encoded by a set of landmarks $(q_i)_{i=1,...,n} \in (\mathbb{R}^d)^n$ linked by edges belonging to $F_q \subset \{1, ..., n\}^2$. Then, the action of vector fields on discrete curve is defined by the action on landmarks $v \cdot q_i = v(q_i)$. The covectors associated with the curve $q$ are represented by vectors $p_i \in \mathbb{R}^d$ with base point $q_i$. Then, the hamiltonian can be expressed as a discrete sum.

$$H(R, T, \tilde{q}, \tilde{p}^A, \tilde{p}^\tau, \tilde{p}) = \frac{1}{2}\left( |\tilde{p}^A R^\top + \tilde{p}^\tau T^\top|^2 + |\tilde{p}^\tau|^2 + \sum_{i,j} \tilde{p}_i^\top K(\tilde{q}_i, \tilde{q}_j)\tilde{p}_j \right) \tag{7.31}$$

In this framework, the constraints can be expressed as a sum over the landmarks

$$\begin{cases} \mu_A = 0 \iff \sum_i \left(\tilde{p}_i \tilde{q}_i^\top - \tilde{q}_i \tilde{p}_i^\top\right) = 0 \\ \mu_\tau = 0 \iff \sum_i \tilde{p}_i = 0 \end{cases} \tag{7.32}$$

The covector $p$ has to be orthogonal to the curve to preserve the invariance by reparameterization, which can be expressed by projecting the covectors on the orthogonal of the curve $\tilde{p} \mapsto \tilde{p} - \langle \tilde{p}, \tilde{q} \rangle \tilde{q}$. Therefore, the conditions on covectors can be enforced by projecting them onto the null space

$$\left\{ (\tilde{p}_i)_{i=1}^n \in (\mathbb{R}^d)^n \;\middle|\; \sum_{i=1}^n (\tilde{p}_i \tilde{q}_i^\top - \tilde{q}_i \tilde{p}_i^\top) = 0, \; \sum_{i=1}^n \tilde{p}_i = 0, \text{ and } \langle \tilde{q}_i, \tilde{p}_i \rangle = 0 \text{ for all } i \in [\![1, n]\!] \right\} \tag{7.33}$$

However, we will often not impose the conditions $\langle \tilde{q}_i, \tilde{p}_i \rangle = 0$ required for invariance by reparameterization, but will instead use a varifold data attachment that implicitly incorporates them [21].

**Remark 7.28.** *If we want to enforce the condition on rotations only, $\mu_A = 0$ can be satisfied by projecting the covectors $p_i$ onto the null space of the functional $F : p \mapsto \sum_i p_i q_i^\top - q_i p_i^\top$ using a SVD decomposition. On the other hand, to enforce the condition on translation only $\mu_\tau = 0$, we can center the covectors $p - \sum_i p_i$. Due to Noether's theorem, these conditions have to be satisfied at time t=0 only and then will remain equal to zero during the motion.*

**Results** We propose in this part numerical experiments illustrating this coupled registration. Figure 7.1 shows source curve (blue) and a target (red) that we want to match. The purpose of this experiment is to perform a matching task between the shapes through a deformation generated by a diffeomorphic deformation and a rotation. The optimization problem is thus

$$\inf_{(A,v) \in L^2([0,1], \mathfrak{so}_d \times V)} J(A, v) = \int_0^1 \frac{1}{2}|A_t|^2 + \frac{1}{2}|v_t|^2 \, dt + \mathcal{D}(R_1 \tilde{q}_1) \tag{7.34}$$

$$\text{s.t} \quad \begin{cases} \dot{R}_t = A_t R_t \\ \dot{\tilde{q}}_{i,t} = R_t^{-1} v_t(R_t \tilde{q}_{i,t}) \quad \text{for } i \in \{1, ..., n\} \\ (R_0, \tilde{q}_0) = (I_d, q_S) \end{cases}$$

where $\tilde{q} = R^{-1}q$. We consider a varifold data attachment term $\mathcal{D}$ since it is particularly well-adapted to the matching of curves due to the invariance by reparameterization [21]. Ideally, the rotation part of the model would encode the entire rotation between the source and the target, while the deformation would capture all other deformations.

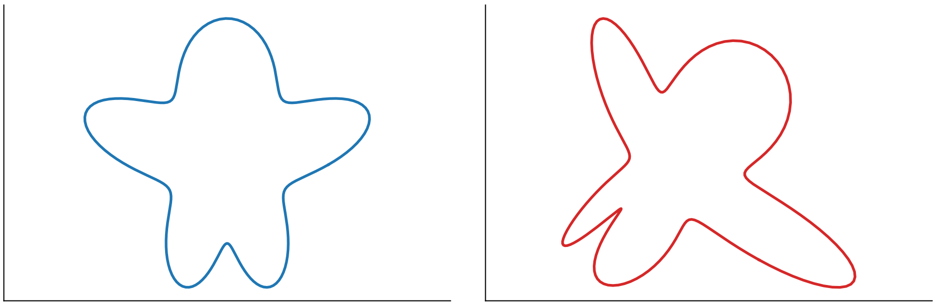

Figure 7.1: **Performing jointly rigid and non rigid registration.** Source shape (in blue), and target shape (in red). The target consists in a rotation of the source, in addition of a small diffeomorphic deformation

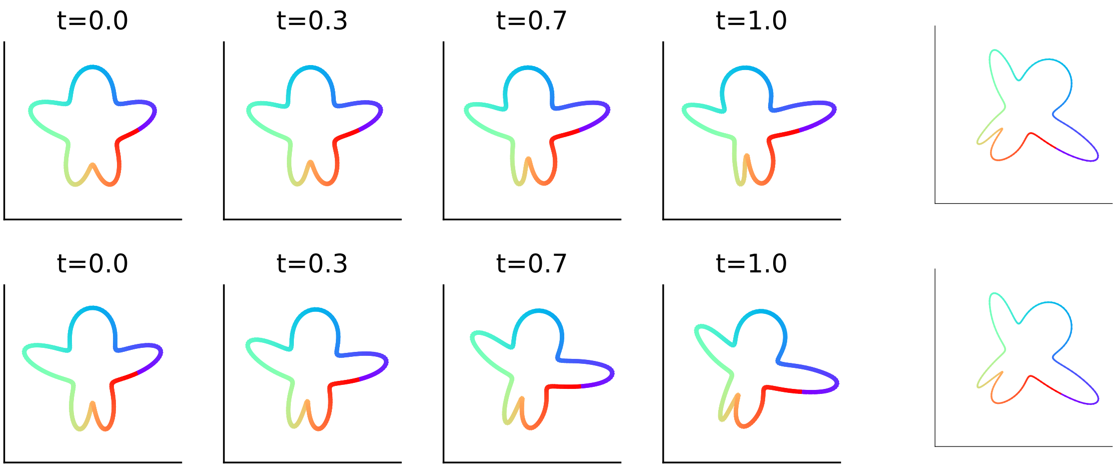


Figure 7.2: **Influence of the constraints on geodesics**. This experiment illustrates a matching task between the source (blue) and the target (red) of Figure 7.1. The matching is performed through deformations generated by rotations and diffeomorphic deformation induced by a gaussian RKHS, solving problem (7.34). Top row corresponds to a registration with the constraint $\mu_A = 0$ enforced at time $t = 0$. Bottom row corresponds to the same without the constraint enforced. For both experiments, we show the evolution of $\tilde{q}_t$, in order to represent only the deformations induced by the diffeomorphisms. The right column corresponds to the final matching where we apply both the rotation and the diffeomorphism

Figure 7.2 compares the effect of the constraint $\mu_A = 0$ (7.32) enforced at time $t = 0$. This first example illustrates that when the constraints are not enforced, it leads to an uncomplete and less stable matching, since the diffeomorphic deformation can learn the rotation too. On the contrary, we do not encounter this problem when the constraints are enforced. Yet, a better choice of the parameters generating the vector fields and defining the cost, could lead to a perfect matching even when the constraints are not enforced : we want to emphasize the fact that the constraints here reduce the space of possible non-rigid deformations, and thereby facilitate the convergence of the LDDMM component. On the contrary, when the constraints are not enforced, convergence toward the non-rigid deformation is slower.

Figure 7.3 shows several geodesics, with the constraint $\mu_A = 0$ enforced, starting from the same source on which we have applied different rotations before the optimization. It illustrates that the constraint allows a total decoupling of the diffeomorphic deformation and the rotation. Indeed those three optimizations lead to the same diffeomorphic deformations, and only the optimized rotation differs.

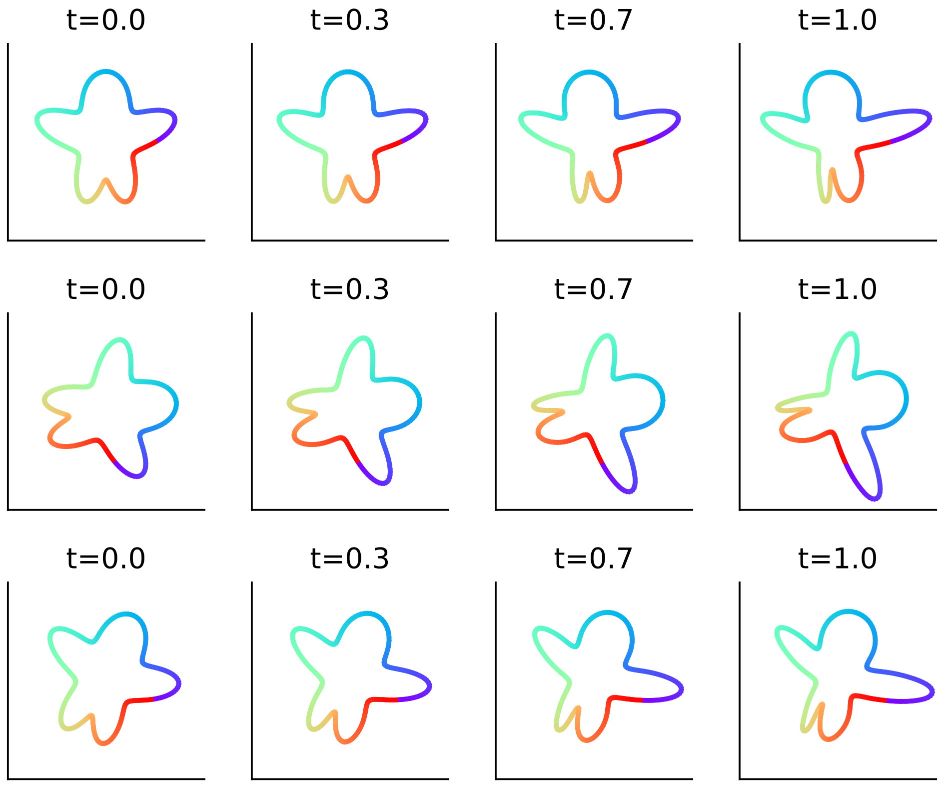


Figure 7.3: **Geodesics starting from the source rotated**. This experiment illustrates the same matching task (7.34) with the same target, but we apply to the source different rotations before the optimization. The constraint $\mu_A = 0$ is still enforced. Each row shows the evolution of $\tilde{q}_t$ in (7.34) starting from the rotated source.

## 7.3 Multiscale matching : hierarchical scheme

We introduce in this section a hierarchical scheme allowing a multiscale registration of shape spaces. We adapt in particular the ideas of [71], where the authors introduce a product group of diffeomorphisms groups, and a right-invariant sub-Riemannian metric on this group allowing interactions along the different scales. We will also explain how this setting differs from a simple sum of kernels as developed in [18, 84, 89], in particular through the definition of a data attachment term. Moreover we also add on this product group the group of rotations and translations, allowing rigid motions as a first coarse layer. Here the coupling of rigid and non-rigid motions will be a bit different from the previous section 7.2, since we will allow rotations around the barycenter of the shapes. In order to deal with this new action, we define this rigid action on the target shape, while the diffeomorphic part will act on the source.

### 7.3.1 First model : sum of kernels

In this part, we define a first multi-scale version of Large Diffeormorphic Metric Mapping (LDDMM). We first follow approaches described in [18, 84, 89], where the authors used a sum of Gaussian kernel in order to transport the shape at different scale simultaneously. Let $\boldsymbol{\mathcal{Q}} = \prod_{1\leq l\leq L} \mathcal{Q}_l$ be a Banach manifold, with $L \geq 0$ and where each $\mathcal{Q}_l$ is also a Banach manifold representing the shape at the scale $l$, $l = 0$ being the coarsest scale and $l = L$ the finest. We recall the definition of the group of diffeomorphisms

$$\mathrm{Diff}_{C_0^k}(\mathbb{R}^d) = \big(\mathrm{id} + C_0^k(\mathbb{R}^d, \mathbb{R}^d)\big) \cap \mathrm{Diff}^1(\mathbb{R}^d)$$

where $C_0^k(\mathbb{R}^d, \mathbb{R}^d)$ is the space of $C_0^k$ mapping whose derivatives up to order $k$ are vanishing at infinity. This is a half Lie group and the family of groups $\{\mathrm{Diff}_{C_0^k}(\mathbb{R}^d), k \geq 1\}$ satisfies the conditions (G.1-5) 4.3. We suppose that the group $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ acts on each layer $\mathcal{Q}_l$ verifying (S.1-3) conditions. This allows to transport the shape at each scale and define a first multi-scale version of Large Diffeormorphic Metric Mapping (LDDMM). We therefore introduce a family of kernels $(K^l)_{1\leq l\leq L}$ for each scale and define the space of controls as a reproducible kernel Hilbert space (RKHS) for the associated kernel $\sum_l K^l$. This allows to consider the multiscale matching problem

$$\inf_{v^1,\dots,v^L} J(v^1, .., v^l) = \frac{1}{2}\sum_{l=1}^{L} \int_I |v^l|^2_{V_l}\, dt + \mathcal{D}(\varphi_1 \cdot q_0^L) \tag{7.35}$$

$$\text{s.t} \quad \begin{cases} \varphi_0 = \mathrm{id} \\ \dot{\varphi}_t = (\sum_{l=1}^{L} v^l) \circ \varphi_t \end{cases}$$

where we suppose that we have a sequence of continuous embeddings $V_1 \hookrightarrow \cdots V_{L-1} \hookrightarrow V_L \hookrightarrow C_0^k(\mathbb{R}^d)$ associated with each kernel $K^l$. Let $\boldsymbol{V} = \prod_l V_l$ be the product Hilbert space of controls equipped with the Hilbert norm defined by $|\mathbf{v}|^2_{\boldsymbol{V}} = \sum_{l=1}^{L} |v^l|^2_{V_l}$ for $\mathbf{v} = (v^l)_{1\leq l<L} \in \boldsymbol{V}$. To deal with problem (7.35), we introduce another Hilbert norm on $\boldsymbol{V}$ given by $|\boldsymbol{u}|_A = |A\boldsymbol{u}|_{\boldsymbol{V}}$ where $A : \boldsymbol{V} \to \boldsymbol{V}$ is a linear isomorphism such that

$$A\boldsymbol{u} = (u^1, u^2 - u^1, \dots, u^L - u^{L-1}).$$

We also introduce the family of groups

$$\mathbf{G}^k = \prod_{1\leq l\leq L} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$$

and the matching problem (7.35) is equivalent to study normal sub-Riemannian geodesics induced by the norm $|\cdot|_A$ on $\mathbf{G}^k$. Note also that the data attachment term depends only on the finest scale $q^L$, so that we can define the associated pre-Hamiltonian only on $\mathcal{Q}_L$ by

$$\mathcal{H}_{\mathcal{Q}_L}(q, p, \boldsymbol{u}) = \big(p \mid \xi_q(u^L)\big) - \frac{1}{2}|A\boldsymbol{u}|^2_{\boldsymbol{V}} \tag{7.36}$$

We get the following

**Proposition 7.29.** *Let $t \mapsto (q(t), p(t), \boldsymbol{u}(t))$ be a normal geodesic associated with the Hamiltonian $\mathcal{H}_{\mathcal{Q}_L}$, and let $\mathbf{v} = A\boldsymbol{u}$. Then we get*

$$v^l = K^l\left(\xi_q^* p\right)$$

*and*

$$u^L = \sum_{l \leq L} K^l\left(\xi_q^* p\right)$$

*Proof.* The proof is mainly included in [18], but we recall it for sake of completeness. We compute $\partial_{\boldsymbol{u}}\mathcal{H}_{\mathcal{Q}_L}(q, p, \boldsymbol{u})$. We denote $L^l = (K^l)^{-1}$ for $1 \leq l \leq L$, and $L_{\boldsymbol{V}} : \boldsymbol{V} \to \boldsymbol{V}'$ the Riesz canonical isometry. Let $\delta\boldsymbol{u} \in \boldsymbol{V}$ we get :

$$\begin{aligned}
\partial_{\boldsymbol{u}}\mathcal{H}_{\mathcal{Q}_L}(q, p, \boldsymbol{u})\,\delta\boldsymbol{u} &= \left(p \mid \xi_q(\delta u^L)\right) - (L_{\boldsymbol{V}} A\boldsymbol{u} \mid A\delta\boldsymbol{u}) \\
&= \left(p \mid \xi_q(\delta u^L)\right) - \sum_{l=2}^{L}(L^l v^l \mid \delta u^l - \delta u^{l-1}) - (L^1 v^1 \mid \delta u^1) \\
&= \left(p \mid \xi_q(\delta u^L)\right) - \sum_{l=1}^{L-1}(L^l v^l - L^{l+1}v^{l+1} \mid \delta u^l) - (L^L v^L \mid \delta u^L)
\end{aligned}$$

In particular, the optimality condition $\partial_u\mathcal{H}_{\mathcal{Q}_L}(\delta\boldsymbol{u}) = 0$ implies

$$\begin{cases} L^l v^l - L^{l+1}v^{l+1} = 0, \ 1 \leq l \leq L-1 \\ \xi_q^* p = L^L v^L \end{cases}$$

and therefore $v^l = K^l \xi_q^* p$ for all $1 \leq l \leq L$. As $u^L = \sum_{l=1}^{L} v^l$, we finally get the result. $\square$

In this framework, only the finest scale is controlled since we only get one covector $p$ associated with this finest scale. Yet, one could also prefer to control all scales in order to adopt a more "coarse-to-fine" approach. More recently in [71], the authors study the case where the dynamic for all scales is controlled, i.e. the term $\mathcal{D}(\varphi_1 \cdot q_0^L)$ is now replaced by $\boldsymbol{\mathcal{D}}(\boldsymbol{\varphi_1} \cdot \boldsymbol{q_0})$ with $\boldsymbol{q_0} = (q_l)_l \in \boldsymbol{\mathcal{Q}}$, and where $\boldsymbol{\mathcal{D}} : \boldsymbol{\mathcal{Q}} \to \mathbb{R}$ is given as a sum:

$$\boldsymbol{\mathcal{D}}(\boldsymbol{q}) = \sum_{l \leq L} \mathcal{D}_l(q_l)$$

for any $\boldsymbol{q} = (q_l)_{l \leq L} \in \boldsymbol{\mathcal{Q}}$. This leads to the new variational problem

$$\begin{aligned}
\inf_{v^1,\ldots v^L} J(v^1, \ldots, v^l) &= \frac{1}{2}\sum_{l=1}^{L}\int_I |v^l|_{V_l}^2 dt + \boldsymbol{\mathcal{D}}(\boldsymbol{\varphi_1} \cdot \boldsymbol{q_0}) \\
\text{s.t} \quad &\begin{cases} \varphi_0^1 = \ldots = \varphi_0^L = \mathrm{id} \\ \dot{\varphi}_t^l = (\sum_{k=1}^{l} v^k) \circ \varphi_t^l \quad \forall l \leq L \end{cases}
\end{aligned} \tag{7.37}$$

This can be simply reformulated as

$$\begin{aligned}
\inf_{\boldsymbol{u} \in \boldsymbol{V}} J(\boldsymbol{u}) &= \int_I |A\boldsymbol{u}|_{\boldsymbol{V}}^2 + \boldsymbol{\mathcal{D}}(\boldsymbol{\varphi_1} \cdot \boldsymbol{q_0}) \\
\text{s.t} \quad &\begin{cases} \boldsymbol{\varphi} = (\mathrm{id}, \ldots, \mathrm{id}) \\ \dot{\boldsymbol{\varphi}} = \boldsymbol{u} \circ \boldsymbol{\varphi} \end{cases}
\end{aligned} \tag{7.38}$$

We get for $\boldsymbol{p} = (p^l)_{1\leq l\leq L}$ and $\boldsymbol{q} = (q^l)_{1\leq l\leq L}$ the corresponding Hamiltonian

$$\mathcal{H}_{\boldsymbol{Q}}(\boldsymbol{q},\boldsymbol{p},\boldsymbol{u}) = \sum_{l=1}^{L} \left(p^l \mid \xi_{q^l}(u^l)\right) - \frac{1}{2}|A\boldsymbol{u}|^2_{\boldsymbol{V}}\,. \tag{7.39}$$

and therefore the minimizers of problem 7.38 satisfies the Hamiltonian equations :

**Proposition 7.30.** *Let $t \mapsto (\boldsymbol{q}(t), \boldsymbol{p}(t), \boldsymbol{u}(t))$ be a normal geodesic associated with the Hamiltonian $\mathcal{H}_{\boldsymbol{Q}}$, and let $\mathbf{v} = A\boldsymbol{u}$. Then we get*

$$v^l = K^l\left(\textstyle\sum_{m\geq l}\xi^*_{q^m}p^m\right) \quad \text{and} \quad u^l = \textstyle\sum_{k\leq l}K^k\left(\textstyle\sum_{m\geq k}\xi^*_{q^m}p^m\right)$$

*Proof.* The proof is similar to the proof of proposition 7.29 and follows from equality $\partial_{\boldsymbol{u}}\mathcal{H}_{\boldsymbol{Q}} = 0$, where

$$\partial_{\boldsymbol{u}}\mathcal{H}_{\boldsymbol{Q}}(\boldsymbol{q},\boldsymbol{p},\boldsymbol{u})\,\delta\boldsymbol{u} = \sum_{l=1}^{L}\left(p^l \mid \xi_{q^l}(\delta u^l)\right) - \sum_{l=2}^{L}(L^l v^l \mid \delta u^l - \delta u^{l-1}) - (L^1 v^1 \mid \delta u^1)$$

□

**Remark 7.31.** *Originating from [71], the aforementioned setting could be also compared to another multiscale approach coupling hierarchical multiscale image and diffeomorphism decomposition in a registration setting [90, 72, 26]. In the later framework, the approach relies on a greedy sequential decomposition of residuals from coarse to fine scales. The resulting output is a deformation achieved by the composition of a sequence of diffeomorphisms. This approach effectively integrates image and diffeomorphism decomposition within a well-defined setting specifically designed from image registration. However, it does not align with the sub-riemannian approach since the scales are considered sequentially in time rather than in parallel. The simultaneous action of all the scales through time is a distinctive feature of the setting* (7.37) *and* (7.38) *that allows its integraion into the (GGA) framework and enables the production of time homogeneous trajectories at different resolutions.*

### 7.3.2 A hierarchical variational problem

In classical LDDMM, a first step is often performed to align source and target objects through rigid motions. In the multiscale setting, one can actually add rigid motion alignment as a coarse first layer. We introduce the group of rotations and translations

$$\mathrm{Rigid}(\mathbb{R}^d) = \mathbb{R}^d \times SO(d)$$

Here we do not consider the usual semidirect product group $\mathrm{Isom}(\mathbb{R}^d) = SO_d \ltimes \mathbb{R}^d$, but we introduce the component wise law :

$$(R,T)\cdot(R',T') = (RR', T+T')$$

Indeed, usually affine isometries are considered as deformations that act on $\mathbb{R}^d$, $(R,T)$ : $x \in \mathbb{R}^d \mapsto Rx + T$. Here we want to include rotations around the barycenter and we thus change the action. Indeed, the group $\mathrm{Rigid}(\mathbb{R}^d)$ naturally acts on the group of landmarks through

$$(R,\tau)\cdot(q_i)_{i\le m} = (R(q_i - q_c) + q_c + T)_{i\le m}$$

where $(q_i) \in (\mathbb{R}^d)^m$, and $q_c = \frac{1}{m}\sum_{i\le m} q_i$ is the barycenter of $(q_i)$.

We suppose the Lie group $\mathrm{Rigid}(\mathbb{R}^d)$ acts on a Banach manifold $\mathcal{Q}_0$ considered as the coarsest layer. We suppose the action is smooth and can be extended on each layer $\mathcal{Q}_l$, $0 \le l \le L$. We can therefore introduce the new group $\boldsymbol{K}^k$ including rigid motions

$$\boldsymbol{K}^k = \mathrm{Rigid}(\mathbb{R}^d) \times \prod_{1\le l\le L} \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$$

**Proposition 7.32.** *The family of groups $\{\boldsymbol{K}^k, k\in\mathbb{N}\}$ satifies conditions (G.1-5) p.50*

*Proof.* The result is straightforward, since $\mathrm{Diff}_{C_0^k}$ satisfies conditions (G.1-5), and $\mathrm{Rigid}(\mathbb{R}^d)$ is finite dimensional Lie groups so immediately satisfies conditions (G.1-5) (here the family would just consists in one group $\{\mathrm{Rigid}(\mathbb{R}^d)\}$). The result follows immediately as $\boldsymbol{K}^k$ has component-wise law of composition □

We now want to adapt the ideas of previous section 7.3.1 and include the group $\mathrm{Rigid}(\mathbb{R}^d)$ to perform a coarse-to-fine matching adding a rigid motion. Therefore we define a right-invariant sub-Riemannian structure on $\boldsymbol{K}^k$. However in this case the Lie algebra $\mathfrak{rigid}(\mathbb{R}^d) = \mathbb{R}^d \times \mathfrak{so}_d$ is not necessarily embedded in the Hilbert space $V_1$, and we cannot directly adapt the idea from the previous section of a hierarchical scheme. To bypass the problem, we separate the actions of $\mathrm{Rigid}(\mathbb{R}^d)$ and of the diffeomorphisms and we introduce an action of $\boldsymbol{K}^k$ on $\boldsymbol{\mathcal{Q}}\times\boldsymbol{\mathcal{Q}}$ where $\boldsymbol{\mathcal{Q}} = \prod_{0\le l\le L}\mathcal{Q}_l$ :

$$\big((\tau,R),(\varphi^l)_{l\ge1}\big)\cdot\big((q_S^l)_l,(q_T^l)_l\big) = \big(\left(\varphi^l\cdot q_S^l\right)_l, \left((\tau,R)\cdot q_T^l\right)_l\big) \tag{7.40}$$

where $\varphi^0 = \mathrm{Id}$, and $(q_S^l)_l$ is viewed as the source object, and $(q_T^l)$ the target. Intuitively, we match the source and target applying rigid motions to the target and finer deformations on the source.

**Proposition 7.33.** *The action of $\boldsymbol{K}^k$ on $\boldsymbol{\mathcal{Q}}\times\boldsymbol{\mathcal{Q}}$ satisfies (S.1-4).*

*Proof.* Once again, the proof is straightforward, since $\mathrm{Rigid}(\mathbb{R}^d)$ acts smoothly on $\boldsymbol{\mathcal{Q}}$, and smooth Lie group actions automatically satify (S.1-4). □

We can now define the matching problem including the rigid deformations :

$$\inf_{(s,\boldsymbol{u})\in\mathfrak{isom}(\mathbb{R}^d)\times\boldsymbol{V}} \frac{1}{2}\int_I \left(|s|^2_{\mathfrak{rigid}(\mathbb{R}^d)} + |\boldsymbol{u}|_A^2\right) dt + \mathcal{D}\left((S,\boldsymbol{\varphi_1})\cdot(\boldsymbol{q_S},\boldsymbol{q_T})\right) \tag{7.41}$$

with dynamic given by

$$\begin{cases} \dot{S}_t = s_t\cdot S_t \\ \dot{\varphi}_t^l = (\sum_{k=1}^l v^k)\circ\varphi_t^l \quad \forall l\le L \end{cases}$$

with $s_t \in \mathfrak{rigid}(\mathbb{R}^d)$. We can therefore introduce the Hamiltonian of the optimal control problem

$$\begin{aligned}\mathcal{H}_{\boldsymbol{\mathcal{Q}}\times\boldsymbol{\mathcal{Q}}}\big((\boldsymbol{q_S},\boldsymbol{q_T}),(\boldsymbol{p_S},\boldsymbol{p_T}),(s,\boldsymbol{u})\big) &= (\boldsymbol{p_S}|\boldsymbol{u}\cdot\boldsymbol{q_S}) + (\boldsymbol{p_T}|s\cdot\boldsymbol{q_T}) - \frac{1}{2}\left(|s|^2_{\mathfrak{rigid}(\mathbb{R}^d)} + |\boldsymbol{u}|_A^2\right)\\ &= \sum_{l=1}^L \left(p_S^l|\xi_{q^l}(u^l)\right) + \sum_{l=0}^L\left(p_T^l|s\cdot q_T^l\right) - \frac{1}{2}\left(|s|^2_{\mathfrak{rigid}(\mathbb{R}^d)} + |A(u^l)_l|_V^2\right)\end{aligned}$$

with $\boldsymbol{q_S} = (q_S^l)_l, \boldsymbol{q_T} = (q_T^l)_l \in \boldsymbol{\mathcal{Q}}$, $\boldsymbol{p_S} = (p_S^l)_l, \boldsymbol{p_T} = (p_T^l)_l \in \boldsymbol{\mathcal{Q}}^*$ and $\boldsymbol{u} = (u^l)_{1\le l\le L} \in \prod_{1\le l\le L} T_e G^k$. Equivalently, we can assimilate $\boldsymbol{q_T}$ with its rigid motions $(R,\tau)$ and get the equivalent Hamiltonian :

$$\begin{aligned}
&\mathcal{H}_{\boldsymbol{\mathcal{Q}}\times\boldsymbol{\mathcal{Q}}}\big((\boldsymbol{q_S},(R,\tau)),(\boldsymbol{p_S},(p_\tau,p_R)),((r,\tau),\boldsymbol{u})\big)\\
&\qquad = (\boldsymbol{p_S} \mid \boldsymbol{u}\cdot\boldsymbol{q_S}) + ((p_\tau,p_R) \mid (r,\tau)\cdot(R,T)) - \frac{1}{2}\left(|r,\tau|^2_{\mathfrak{rigid}(\mathbb{R}^d)} + |\boldsymbol{u}|^2_A\right)\\
&\qquad = \sum_{l=1}^{L}\left(p_S^l \mid \xi_{q_S^l}(u^l)\right) + (p_\tau \mid \sigma) + (p_R \mid rR) - \frac{1}{2}\left(|r,\tau|^2_{\mathfrak{rigid}(\mathbb{R}^d)} + |A(u^l)_l|^2_V\right)
\end{aligned}$$

#### 7.3.2.1 The example of landmarks

We apply the previous approach on the space of landmarks. We define the multiscale space as

$$\boldsymbol{\mathcal{Q}} = \prod_{0\le l\le L}(\mathbb{R}^d)^{I_l}$$

with $(I_l)_l$ an increasing sequence of intervals so that each $(\mathbb{R}^d)^{I_l}$ represents a summary of the data at the scale $l$. The diffeomorphisms act in the following classical action :

$$\varphi^l\cdot(q_i^l)_{i\in I_l} = \left(\varphi^l(q_i^l)\right)_{i\in I_l} \text{ for } l\ge 1$$

We also define the group action of Rigid($\mathbb{R}^d$) on $(\mathbb{R}^d)^{I_l}$ as

$$(R,T)\cdot q_i^l = R(q_i^l - q_c^l) + q_c^l + T \text{ for } i\in I_l$$

with $q_c^l = \frac{1}{|I_l|}\sum_{i\in I_l} q_i^l$ the center of mass. We suppose Rigid($\mathbb{R}^d$) is equipped with the standard right-invariant metric

$$\langle(r,\tau),(r',\tau')\rangle = -\operatorname{tr}(rr') + \langle\tau,\tau'\rangle$$

where $\tau,\tau'\in\mathbb{R}^d$, and $r,r'\in\mathcal{A}_d = T_{\mathrm{id}}SO(d)$ skew-symmetric matrices We take a $L^2$ endpoint constraints

$$\mathcal{D}(\boldsymbol{q},\boldsymbol{q'}) = \frac{1}{2}\sum_{l=0}^{L}\sum_{i\in I_l}|q_i^l - q_i'^l|^2.$$

We also introduce the pre-Hamiltonian associated with the variational problem 7.41, with the data attachment term $\mathcal{D}$

$$\begin{aligned}
\mathcal{H}_{lmk}\left((\boldsymbol{q},(R,T)),(\boldsymbol{p},(p_\tau,p_R)),(r,\tau),\boldsymbol{u})\right) = \sum_{l=1}^{L}\sum_{i\in I_l}\left(p_i^l \mid u^l(q_i^l)\right) + (p_\tau \mid \tau) + (p_R \mid rR)\\
- \frac{1}{2}\left(|r,\tau|^2_{\mathfrak{rigid}(\mathbb{R}^d)} + |A(u^l)_l|^2_V\right)
\end{aligned} \tag{7.42}$$

We recall the solutions of the variational problem 7.41 satisfies the hamiltonian equations

$$\begin{cases}
\left(\dot{\boldsymbol{q}},(\dot{R},\dot{T})\right) = \partial_p\mathcal{H}_{lmk}\left((\boldsymbol{q},(R,T)),(\boldsymbol{p},(p_\tau,p_R)),((r,\tau),\boldsymbol{u})\right)\\
(\dot{\boldsymbol{p}},(\dot{p_\tau},\dot{p_R})) = -\partial_q\mathcal{H}_{lmk}\left((\boldsymbol{q},(R,T)),(\boldsymbol{p},(p_\tau,p_R)),((r,\tau),\boldsymbol{u})\right)\\
\partial_{\sigma,r,\boldsymbol{u}}\mathcal{H}_{lmk}\left((\boldsymbol{q},(R,T)),(\boldsymbol{p},(p_\tau,p_R)),((r,\tau),\boldsymbol{u})\right) = 0
\end{cases}$$

**Proposition 7.34.** *Let $\boldsymbol{q}_S, \boldsymbol{q}_T \in \boldsymbol{\mathcal{Q}}$ source and target objects. Then the solution $((\boldsymbol{q}_t, R_t, \tau_t), (\boldsymbol{p}_t, p_{\tau,t}, p_{R,t}))$ of the variational problem 7.41 starting from $\boldsymbol{q}_S$ satisfies*

$$\begin{cases} \dot{q}^l_{i,t} = u^l_t(q_{i,t}), \ l \geq 1 \\ \dot{p}^l_{i,t} = -\left(du^l_t(q^l_{i,t})\right)^* p^l_{i,t} \\ \dot{T}_t = \tau, \dot{R}_t = rR_t \\ (\dot{p}_{\tau,t}, \dot{p}_{R,t}) = (0, -r^* p_{R,t}) \end{cases}$$

*with $\tau \in \mathbb{R}^d$, $r \in \mathcal{A}_d = T_{\mathrm{id}}SO(d)$, and $\boldsymbol{u}_t$ satisfying*

$$u^l_t = \sum_{k \leq l} K^k \left( \sum_{m \geq k} \sum_{i \in I_m} p^m_{i,t} \circ \delta_{q^m_{i,t}} \right)$$

*Moreover, we have the following endpoint conditions for the coadjoint map*

$$\begin{cases} p^l_{i,1} = -q^l_{i,1} + R_1(q^l_{T,i} - q^l_{T,c}) + q^l_{T,c} + T_1, \ l \geq 1 \\ p_{R,1} = -\frac{1}{2} \sum_{l=0}^L \sum_{i \in I_l} p^l_{i,1}(q^l_{T,i} - q^l_{T,c})^T - R_1(q^l_{T,i} - q^l_{T,c}) {p^l_{i,1}}^T R_1 \\ p_{\tau,1} = -\sum_{l=0}^L \sum_{i \in I_l} p^l_{1,i} \end{cases} \tag{7.43}$$

*where $p^0_{1,i} = -q^l_{S,i} + R_1(q^l_{T,i} - q^l_{T,c}) + q^l_{T,c} + T_1$ corresponds to coadjoint variable associated with rigid motions.*

**Remark 7.35.** *In this case, the variable $\boldsymbol{q}$ codes the evolution from source object $q_S$ to target object $q_T$. Since the rigid motion is only applied to $q_T$, the coarse layer of $\boldsymbol{q}$ is just constant and verifies $q^0_{i,t} = q_{S,i}$*

*Proof.* The differential equations $\dot{q}^l_{i,t} = u^l_t(q_{i,t})$ and $\dot{p}^l_{i,t} = -\left(du^l_t(q^l_{i,t})\right)^* p^l_{i,t}$ follow directly from the computation of the derivatives $\partial_p \mathcal{H}_{lmk}$ and $\partial_q \mathcal{H}_{lmk}$. Furthermore, similarly to propositions 7.35 and 7.29, solving $\partial_{\boldsymbol{u}} \mathcal{H}_{lmk} = 0$ gives us

$$u^l_t = \sum_{k \leq l} K^k \left( \sum_{m \geq k} \sum_{i \in I_m} p^m_{i,t} \circ \delta_{q^m_{i,t}} \right)$$

The equations for the rigid part are

$$\begin{cases} (\dot{R}, \dot{T}) = \partial_{p_\tau, p_R} \mathcal{H}_{lmk} \\ (\dot{p_\tau}, \dot{p_R}) = -\partial_{\tau, R} \mathcal{H}_{lmk} \\ \partial_{\sigma, r} \mathcal{H}_{lmk} = 0 \end{cases}$$

and can be simply written as

$$\begin{cases} (\dot{R}, \dot{T}) = (r_t R_T, \tau_t) \\ (\dot{p_\tau}, \dot{p_R}) = (0, -r^*_t p_{R,t}) \\ p_\tau = \tau_t, \\ \forall \delta r \in \mathcal{A}_d, \ (p_{R,t} \mid \delta r R) = \langle r, \delta r \rangle \end{cases}$$

This gives us in particular that $\tau_t = \tau \in \mathbb{R}$ is constant. Moreover, combining the rotation equations, we have for $\delta r \in \mathcal{A}_d$

$$\begin{aligned}
\langle \dot{r}_t, \delta r \rangle &= \frac{d}{dt}(p_{R,t} | \delta r R_t) \\
&= -(p_{R,t} | r_t \delta r R_t) + (p_{R,t} | \delta r r_t R_t) \\
&= -\langle r_t, r_t \delta r \rangle + \langle r_t, \delta r r_t \rangle \\
&= \operatorname{tr}(r_t^2 \delta r) - \operatorname{tr}(r_t \delta r r_t) \\
&= 0
\end{aligned}$$

which are the classical geodesic equations for the right-invariant metric on $SO(d)$. This also gives that $r_t = r \in \mathcal{A}_d$ is constant.

We finish by proving equations (7.43), which follows from the endpoint conditions. Indeed, for the constraint $\mathcal{D}(\boldsymbol{q}, R, T) = \frac{1}{2} \sum_{l=0}^{L} \sum_{i \in I_l} \left| q_i^l - R(q_{T,i}^l - q_{T,c}^l) - q_{T,c}^l - T \right|^2$, we have

$$(\boldsymbol{p}_1, p_{\tau,1}, p_{R,1}) = -d\mathcal{D}(\boldsymbol{q}_1, R_1, T_1) \tag{7.44}$$

For $l \geq 1$, $i \in I_l$, and $\delta q_i^l \in \mathbb{R}^d$ we then have

$$\begin{aligned}
(p_{i,1}^l \mid \delta q_i^l) &= -\partial_{q_i^l} \mathcal{D}(\boldsymbol{q}_1, R_1, T_1) \delta q_i^l \\
&= -\langle q_{i,1} - R_1(q_{T,i}^l - q_{T,c}^l) - q_{T,c}^l - T, \delta q_i^l \rangle
\end{aligned}$$

and therefore $p_{i,1}^l = -q_{i,1} + R_1(q_{T,i}^l - q_{T,c}^l) + q_{T,c}^l + \tau$. Similarly, for $l = 0$, we also have $p_{i,1}^0 = -q_{S,i}^0 + R_1(q_{T,i}^0 - q_{T,c}^0 l) + q_{T,c}^0 + \tau$. Moreover, for any

$$\delta R \in T_{R_1} SO(d) = \{X \in \mathcal{M}_d,\ X^T R_1 + R_1^T X = 0\},$$

we get

$$\begin{aligned}
(p_{R,1} | \delta R) &= -\partial_R g(\boldsymbol{q}_1, \tau_1, R_1) \delta R \\
&= -\sum_{l \leq L} \sum_{i \in I_l} \langle q_{i,1} - R_1(q_{T,i}^l - q_{T,c}^l) - q_{T,c}^l - \tau, -\delta R(q_{T,i}^l - q_{T,c}^l) \rangle \\
&= -\sum_{l \leq L} \sum_{i \in I_l} (p_{i,1}^l | \delta R(q_{T,i}^l - q_{T,c}^l)) \\
&= -\sum_{l \leq L} \sum_{i \in I_l} \operatorname{tr}\left(p_{i,1}^l (q_{T,i}^l - q_{T,c}^l)^T \delta R\right)
\end{aligned}$$

Since we also have

$$\begin{aligned}
\operatorname{tr}\left((R_1(q_{T,i}^l - q_{T,c}^l) {p_{i,1}^l}^T R_1)^T \delta R\right) &= \operatorname{tr}\left(p_{i,1}^l (q_{T,i}^l - q_{T,c}^l)^T R_1^T \delta R R_1^T\right) \\
&= \operatorname{tr}\left(p_{i,1}^l (q_{T,i}^l - q_{T,c}^l)^T R_1^T \delta R R_1^T\right) \\
&= -\operatorname{tr}\left(p_{i,1}^l (q_{T,i}^l - q_{T,c}^l)^T \delta R\right)
\end{aligned}$$

and since $p_{i,1}^l (q_{T,i}^l - q_{T,c}^l)^T - (R_1(q_{T,i}^l - q_{T,c}^l) {p_{i,1}^l}^T R_1)^T \in T_{R_1} SO(d)$, we obtain

$$(p_{R,1} \mid \delta R) = -\frac{1}{2} \sum_{l \leq L} \sum_{i \in I_l} \left(p_{i,1}^l (q_{T,i}^l - q_{T,c}^l)^T - R_1(q_{T,i}^l - q_{T,c}^l) {p_{i,1}^l}^T R_1, \delta R\right)$$

Finally, for the translation part, we have for $\delta\tau \in \mathbb{R}^d$,

$$\begin{aligned}
(p_\tau \mid \delta\tau) &= -\partial_\tau g(\boldsymbol{q}_1, \tau_1, R_1)\delta\tau \\
&= -\sum_{l \leq L} \sum_{i \in I_l} \langle q_i^l - R(q_{T,i}^l - q_{T,c}^l) - q_{T,c}^l - \tau, -\delta\tau \rangle \\
&= -\sum_{l \leq L} \sum_{i \in I_l} (p_{i,1}^l \mid \delta\tau)
\end{aligned}$$

Hence the proposition. $\square$

# Chapter 8

# Anisotropy I : Anisotropic Gaussian kernel and scaling action

## Contents



This chapter describes the main results of the second part of [74], and is a joint work with PhD student Rayane Mouhli. We tackle here the issue of incorporating anisotropy in the LDDMM setting, and more particularly in its extension we presented in this thesis in chapter 7 (cf. 7.2). The anisotropy of the shape will be seen as extra-information that we also want to transport, in particular by the action of scalings and rotations. We propose two applications where this anisotropy is included either in the metric used to compute the energy of the deformations, or directly as part of the data.

In section 8.1, we follow the ideas presented in section 7.2.5 and assume that the deformations result from a coupling of scaling, isometries and diffeomorphisms. To generate the diffeomorphisms, we use a RKHS induced by an anisotropic Gaussian kernel, meaning that the deformations will favor certain axes. Moreover we also transport this anisotropic metric along the action of the group of deformations, losing the right-invariance of the sub-Riemannian metrics of chapter 6. This allows in particular to follow the global anisotropy of the data during the motion. We apply this model on the space of landmarks, using a hierarchical scheme as in 7.3.

Section 8.2 continues the ideas of 8.1 adding an anisotropic scaling, meaning that the scaling is not the same on each axis. We therefore adapt the group of deformations and propose an application to matching of images. In particular, we recall the specificities of large deformation model on the space of images, notably since the action of diffeomorphisms on images does not fit in the (GGA) framework defined in 6.1 (cf. example 6.4).

## 8.1 Transport of landmarks using anisotropic Gaussian kernels

In this section, we assume that the space $V$ generating vector fields for the diffeomorphism part is a RKHS induced by an anisotropic Gaussian kernel that allows to favor motions along certain privileged axes. Such a kernel is built by replacing the classic euclidian norm in the gaussian kernel by an anisotropic metric encoded by a symmetric definite positive matrix. While deforming shapes, we would like to keep track of the favorized axis encoded by this metric by transporting it along with the shape during the motion (cf. Figures 8.2 and 8.3). To do so, we enrich the shape space by considering the metric as a part of it. Following 7.2 we define the group of deformations as a semidirect product of scalings, isometries and diffeomorphisms, acting on both the metric and the shape encoded as landmarks.

In this setting, we will not obtain a right-invariant sub-Riemannian metric on the group of deformations, since the Gaussian kernel depends now on the metric which is part of the shape.

### 8.1.1 Anisotropic Gaussian kernel

Let $S_d^{++}$ be the space of symmetric definite positive matrices. Any symmetric definite positive matrix $D \in S_d^{++}$ defines an anistropic scalar product $\langle x, y\rangle_D = \langle Dx, y\rangle$ and the associated anisotropic norm $\|q\|_D^2 = \langle q, q\rangle_D$. We define the (normalized) anisotropic Gaussian kernel associated with this metric :

$$k_D(x, y) = \exp\left(-\frac{1}{2}\|x - y\|_{D^{-1}}^2\right) D.$$

and we denote $V_D$ the reproducing kernel Hilbert space associated with $k_D$ and $K_D : V_D^* \to V_D$ the Riesz isomorphism.

Note that in particular, if $D$ is not a multiple of the identity, i.e. if its eigenvalues are distinct, then the kernel $k_D$ is not invariant under rotations anymore, compared to classic isotropic Gaussian kernels (this means that there exists a rotation $R \in SO_d$ such that $k_D(Rx, Ry) \neq k_D(x, y)$). This highlights the fact that the metric $D$ privileges some axis over the other.

#### A first (naive) model

A first naive idea would be to use this Gaussian kernel to perform a coupled deformation of isometries and diffeomorphisms as in section 7.2. However, the introduction of anisotropy in the space $V_D$ through the metric $D$ will make the deformation induced by the diffeomorphism easier along certain directions, leading to a lower energy cost. In particular, the deformation will tend to align the shape along those favorized axes in order to perform these easier deformations. We illustrate this phenomenon with an example on the space of landmarks $\mathrm{Lmk}_n(\mathbb{R}^d)$.

We define, following 7.2.5, the group $SO_d \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ of deformations, allowing to couple rotations and diffeomorphisms. This group acts on the shape space $\mathrm{Lmk}_n(\mathbb{R}^d)$ by

$$(R, \varphi) \cdot (q_i)_{i \leq n} = (R\varphi(q_i))_{i \leq n}$$

and the infinitesimal action becomes $(A, u) \cdot (q_i)_{i \leq n} = (Aq_i + u(q_i))_{i \leq n}$ where $(A, u) \in \mathfrak{so}_d \oplus C_0^k(\mathbb{R}^d, \mathbb{R}^d)$. This allows, given a source shape $\boldsymbol{q}_S = (q_{S,i})$ and a target $\boldsymbol{q}_T = (q_{T,i})$, to introduce the matching problem (cf. general problem 7.15)

$$\inf_{(A,u) \in L^2([0,1], \mathfrak{so}_d \times V_D)} J(A, u) = \frac{1}{2} \int_0^1 |A|^2 + |v_t|_{V_D}^2 \, dt + \mathcal{D}(q_1) \tag{8.1}$$
$$\text{s.t} \quad \begin{cases} \dot{q}_{i,t} = A_t q_{i,t} + u_t(q_{i,t}) \\ q_{i,0} = q_{S,i} \end{cases}$$

where $\mathcal{D} : \mathrm{Lmk}_n \to \mathbb{R}$ is a varifold data attachment term [21, 71] defined by

$$\mathcal{D}(\boldsymbol{q}) = \sum_{i,j} k_W(q_i, q_j) + k_W(q_{T,i}, q_{T,j}) - 2k_W(q_i, q_{T,j})$$

and $k_W$ is a Gaussian kernel (or a sum of Gaussian kernels).

**Results** We test this model by minimizing variational problem 8.1 with a given source and target represented in Figure 8.1. Here the source and target shape are encoded by landmarks with 200 points, and represents elongated shapes along the $y$-axis with the target being larger than the source. We use for the diffeomorphisms an anisotropic gaussian kernel with scales $(\sigma_x, \sigma_y) = (1., 0.1)$, meaning the deformation is easier along the x-axis and harder along the y-axis. We show in 8.2 the geodesic obtained after the minimization of (8.1). We clearly observe that the shape tends to align along the $x$-axis using the rotation part of the deformation, before performing the stretching of the shape necessary to obtain the target with a lower cost. Indeed, the minimization algorithm manages to use the rotation part in order to align the shape with the $x$-axis (favored by the anisotropic kernel) instead of performing a vertical stretching which is the expected deformation. This problem is due to the non-transport of the anisotropy by the rotation. Indeed, the axis of the anisotropy are fixed so the algorithm will tend to first align with the favored axis with the rotation, deforms with the diffeomorphism and then rotating back to the unfavored axis. If the axis of the anisotropy are transported, this trick would not work anymore since they will rotate along with the shape. Note that the alignment with the $x$-axis is not complete and only partial here, this is because the cost for the rotation is not free.

### 8.1.2 Group of deformations

In this section, we define the group of deformations as a semidirect product between the finite dimensional group composed of scalings and isometries, and the group of diffeomorphisms, following section 7.2. As a first step to tackle the issue illustrated in Figure 8.1, we will define an action of this semidirect product on the space of metrics $S_d^{++}$ such that the anisotropic metric is also transported by the group of deformations, in particular through the action of isometries and scalings.

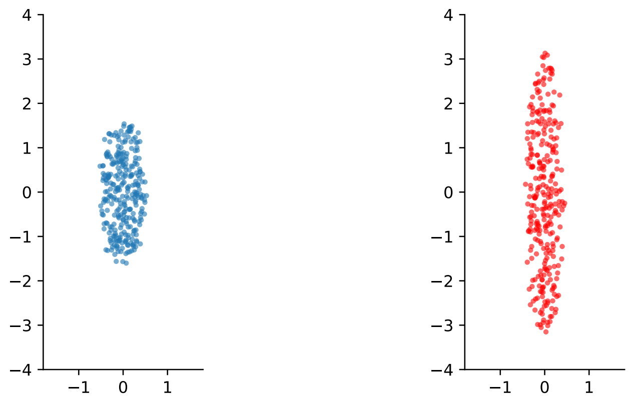

Figure 8.1: *Performing jointly rigid and non rigid registration, with an anisotropic kernel.* Source shape (in blue), 200 points and target shape (in red), 200 points.

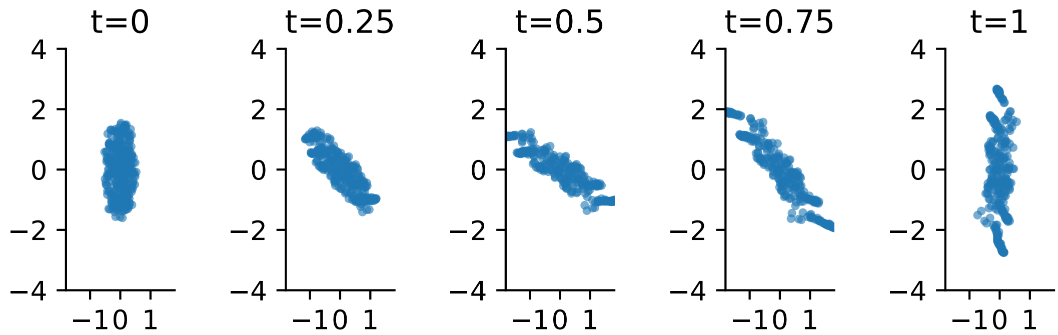


Figure 8.2: *Performing jointly rigid and non rigid registration, minimizing problem 8.1 with an anisotropic kernel.* This experiment illustrates a matching task between the source (blue) and the target (red). The matching is performed through deformations generated by rotations and diffeomorphic deformation induced by a anisotropic gaussian kernel with scale $(\sigma_x, \sigma_y) = (1., 0.1)$.

#### 8.1.2.1 The semidirect product of isometries and scalings

We define here the (finite dimensional) Lie group $\alpha$-Isom$(\mathbb{R}^d) = (\mathbb{R}_{>0} \times SO_d) \ltimes \mathbb{R}^d$ consisting of scalings, rotations and translations, with the composition law

$$(\alpha, R, T) \cdot (\alpha', R', T') = (\alpha\alpha', RR', \alpha RT' + T)$$

The group $\alpha$-Isom$(\mathbb{R}^d)$ acts on $\mathbb{R}^d$ the transformation $\varphi_{(\alpha,R,T)}$ defined by

$$\varphi_{(\alpha,R,T)} : \begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ q & \longmapsto & \alpha Rq + T \end{array}$$

so that

$$\varphi_{(\alpha,R,T)} \circ \varphi_{(\alpha',R',T')} = \varphi_{(\alpha,R,T)\cdot(\alpha',R',T')}.$$

The group of motions $\alpha$-Isom$(\mathbb{R}^d)$ also acts on the left on the space $S_d^{++}$ in the following way

$$(\alpha, R, T) \cdot D = \alpha^2 RDR^\top$$

This action is compatible with the definiton of the anisotropic Gaussian kernel we defined before in the sense we get isometries between the RKHS :

**Proposition 8.1** (Isometries between Gaussian RKHS). *Let $D \in S_d^{++}$, and $u \in V_D$, where $V_D$ is the Gaussian RKHS induced by $D$. Let $(\alpha, R, T) \in \alpha\text{-Isom}(\mathbb{R}^d)$, and define the vector field $u' = d\varphi_{(\alpha,R,T)^{-1}} u \circ \varphi_{(\alpha,R,T)}$ by*

$$\forall q \in \mathbb{R}^d, \quad u'(q) = \alpha^{-1} R^\top u(\alpha R q + T).$$

*Then $u' \in V_{D'}$ with $D' = (\alpha, R, T)^{-1} \cdot D = \alpha^{-2} R^\top D R$ and we get*

$$|u'|_{V_{D'}} = |u|_{V_D}$$

**Remark 8.2.** *In particular this means the map*

$$\begin{array}{ccc} V_D & \longrightarrow & V_{\alpha^{-2}R^\top DR} \\ u & \longmapsto & d\varphi_{(\alpha,R,T)^{-1}} u \circ \varphi_{(\alpha,R,T)} \end{array}$$

*is an isometry of Hilbert spaces.*

*Proof.* Since the family $\{k_D(q, \cdot) b \mid q, b \in \mathbb{R}^d\}$ forms a total subset of the RKHS $V_D$ [11], we restrict to the case $u = \sum_i k_D(q_i, \cdot) b_i$ for some $q_i, b_i \in \mathbb{R}^d$. Let $(\alpha, R, T) \in \alpha\text{-Isom}(\mathbb{R}^d)$, we get for any $q \in \mathbb{R}^d$,

$$\begin{aligned}
& d\varphi_{(\alpha,R,T)^{-1}} u \circ \varphi_{(\alpha,R,T)}(q) \\
&= \sum_i \alpha^{-1} R^\top k_D(q_i, \alpha R q + T) b_i \\
&= \sum_i \exp\big(-\frac{1}{2}\langle D^{-1}(q_i - \alpha R q - T), q_i - \alpha R q - T\rangle\big) \alpha^{-1} R^\top D b_i \\
&= \sum_i \exp\big(-\frac{1}{2}\langle \alpha^2 R^\top D^{-1} R(\alpha^{-1} R^\top (q_i - T) - q), \\
&\qquad \alpha^{-1} R^\top (q_i - T) - q\rangle\big) \alpha^{-2} R^\top D R(\alpha R^\top b_i) \\
&= \sum_i k_{\alpha^{-2} R^\top D R}(\alpha^{-1} R^\top (q_i - T), q) \alpha R b_i
\end{aligned}$$

and thus

$$d\varphi_{(\alpha,R,T)^{-1}} u \circ \varphi_{(\alpha,R,T)} = K_{\alpha^{-2}R^\top DR} \sum_i \delta_{\alpha^{-1}R^\top(q_i - T)}^{\alpha R^\top b_i} \in V_{(\alpha,R,T)^{-1}\cdot D}.$$

Moreover, by definition, we see that

$$\begin{aligned}
& |\varphi_{(\alpha,R,T)^{-1}} u \circ \varphi_{(\alpha,R,T)}|^2_{V_{(\alpha,R,T)^{-1}\cdot D}} \\
&= \sum_{i,j} (\alpha R^\top b_i)^\top k_{\alpha^{-2}R^\top DR}(\alpha R^\top (q_i - T), \alpha R^\top (q_j - T)) \alpha R^\top b_j \\
&= \sum_{i,j} \exp\big(-\frac{1}{2}\langle \alpha^2 R^\top D^{-1} R(\alpha^{-1} R^\top (q_i - q_j)), \\
&\qquad \alpha^{-1} R^\top (q_i - q_j)\rangle\big) \langle \alpha^{-2} R^\top D R(\alpha R^\top b_i), \alpha R^\top b_j\rangle \\
&= \sum_{i,j} \exp\left(-\frac{1}{2}\langle D^{-1}(q_i - q_j)), (q_i - q_j)\rangle\right) \langle D b_i, b_j\rangle \\
&= |u|^2_{V_D}
\end{aligned}$$

which concludes the proof. $\square$

#### 8.1.2.2 Adding the group of diffeomorphisms

Following section 7.2, we can now define the half-Lie group of deformations as a semidirect product of the finite dimensional Lie group $\alpha$-Isom$(\mathbb{R}^d)$ and of the group of $C_0^k$ diffeomorphisms:

$$\alpha\text{-Isom}(\mathbb{R}^d) \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$$

We recall the group operations are given here by

$$\begin{cases} (\alpha, R, T, \varphi) \cdot (\alpha', R', T', \varphi') = \Big(\alpha\alpha', RR', \alpha RT' + T, \varphi^{-1}_{(\alpha', R', T')} \circ \varphi \circ \varphi_{(\alpha', R', T')} \circ \varphi'\Big) \\ (\alpha, R, T, \varphi)^{-1} = \Big(\alpha^{-1}, R^\top, -\alpha^{-1} R^\top T, \varphi_{(\alpha,R,T)} \circ \varphi^{-1} \circ \varphi^{-1}_{(\alpha,R,T)}\Big) \end{cases}$$

and by proposition 7.6 that the family of half-Lie groups $\{\alpha\text{-Isom}(\mathbb{R}^d) \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d),\ k \geq 1\}$ verifies conditions (G.1-5) 4.3.

### 8.1.3 The coupled dynamic on the space of landmarks

We can now define a new model using the group of deformations $\alpha$-Isom$(\mathbb{R}^d) \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ in order to transport the shape and the anisotropy, avoiding the behaviour of Figure 8.2. Note that this model, and the idea of creating interactions between the shapes and the kernel that generates vector fields, is close to the general modular approach for diffeomorphic deformations introduced in [46, 45, 47]. In that work, the authors define a set of geometric descriptor to construct vector fields that transport the shape along with these geometric descriptors via its infinitesimal action. In this section, the global metric inducing the anisotropic Gaussian kernel can be interpreted as one of these geometric descriptors.

In this section, for sake of readiness, we only deal with shapes encoded by landmarks.

#### 8.1.3.1 The shape space

We consider a source shape $q_S \in \mathrm{Lmk}_N(\mathbb{R}^d)$ in the space of landmarks that we want to match to a given target. Moreover, we assume that we have a prior on the distribution of $q_S$ given by a symmetric definite positive matrix $D_S \in S_d^{++}$ representing the anisotropy of the shape. This matrix defines a metric, as stated previously, that we want to transport together with the shape. Consequently, we define in the same fashion as section 7.2.2 the augmented product shape space

$$\mathcal{Q} = \alpha\text{-Isom}(\mathbb{R}^d) \times S_d^{++} \times (\mathbb{R}^d)^N$$

representing the orientation position, and a finer description of the shape with an anisotropic metric associated. The next step is to define the action of the semidirect product $\alpha$-Isom$(\mathbb{R}^d) \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ on the manifold $\mathcal{Q}$. First, the group $\alpha$-Isom$(\mathbb{R}^d)$ acts on the shape space $\mathcal{Q}$ by

$$(\alpha, R, T) \cdot (\alpha', R', T', D, (q_i)_i) = \big(\alpha\alpha', RR', \alpha RT' + T, \alpha^2 RDR^\top, (\alpha R q_i + T)_i\big)$$

with infinitesimal action

$$\xi^{\alpha\text{-Isom}}_{\alpha,R,T,D,q_i}(s, A, \tau) = \big(s\alpha, AR, (s + A)T + \tau, 2sD + AD - DA, sq_i + Aq_i + sT + \tau\big).$$

Moreover the group of diffeomorphisms simply acts on $\mathcal{Q}$ by transporting the points :

$$\varphi \cdot (\alpha', R', T', D, (q_i)_i) = (\alpha', R', T', D, (\varphi(q_i))_i)$$

The combined action of the total group of deformations is thus given by

$$\begin{aligned}((\alpha, R, T), \varphi) \cdot (\alpha', R', T', D, (q_i)_i) &\coloneqq (\alpha, R, T) \cdot \varphi \cdot (\alpha', R', T', D, (q_i)_i) \\ &= \big(\alpha\alpha', RR', \alpha RT' + T, \alpha^2 RDR^\top, (\alpha R\varphi(q_i) + T)_i\big)\end{aligned}$$

Differentiating this expression on the left at identity, we get the infinitesimal action

$$\xi : \left\{\begin{array}{lll}\big(\alpha\text{-}\mathfrak{isom}(\mathbb{R}^d) \times C_0^k(\mathbb{R}^d, \mathbb{R}^d)\big) \times \mathcal{Q} & \longrightarrow & T\mathcal{Q} \\ (s, A, \tau, u), (\alpha, R, T, D, q) & \longmapsto & \big(s\alpha, AR, (s + A)T + \tau, 2sD + AD - DA, \\ & & sq + Aq + u(q) + sT + \tau\big)\end{array}\right. \tag{8.2}$$

#### 8.1.3.2 The matching problem

We suppose we are given a data attachment term $\mathcal{D} : \mathcal{Q} \to \mathbb{R}$. We recall we are given a source shape $\boldsymbol{q_S} \in \mathrm{Lmk}_N(\mathbb{R}^d)$ together with a source metric $D_S \in S_d^{++}$. Note that, since the metric $D_S$ is transported and is used to compute the energy of the vector field transporting the shape, the problem and the sub-Riemannian structure we are studying here is not right-invariant anymore. The induced structure on $\mathcal{Q}$ is given by the bundle $\overline{V}$ over $\mathcal{Q}$ such that for any $(\alpha, R, T, D, q) \in \mathcal{Q}$, we get

$$\overline{V}_{\alpha,R,T,D,q} = \alpha\text{-}\mathfrak{isom}(\mathbb{R}^d) \oplus V_D,$$

where $V_D$ is the RKHS induced by the Gaussian kernel $k_D$, together with the bundle morphism $\xi : \overline{V} \to T\mathcal{Q}$ and the metric defined on $\overline{V}$ by

$$\langle (s, A, \tau, u), (s, A, \tau, u)\rangle_{\alpha,R,T,D,q} = |s|^2 + |A|^2 + |\tau|^2 + |u|^2_{V_D}$$

Moreover, even though this structure is not right-invariant, it depends here only on the metric $D$ and is still invariant by the action of diffeomorphism part. We can define, following 7.2.3 the variational problem

$$\inf_{(s_t,A_t,\tau_t,u_t) \in L^2([0,1],\overline{V})} J(s_t, A_t, \tau_t, u_t) = \frac{1}{2}\int_0^1 |s_t|^2 + |A_t|^2 + |\tau_t|^2 + |u_t|^2_{V_{D_t}}\, dt \tag{8.3}$$
$$+\mathcal{D}(\alpha_1, R_1, T_1, D_1, \boldsymbol{q}_1)$$
$$\text{s.t} \quad \left\{\begin{array}{l}(\dot\alpha_t, \dot R_t, \dot T_t, \dot D_t, \dot{\boldsymbol{q}}_t) = \xi_{\alpha_t,R_t,T_t,D_t,\boldsymbol{q}_t}(s_t, A_t, \tau_t, u_t) \\ (\alpha_0, R_0, T_0, D_0, \boldsymbol{q}_0) = (1, Id, 0, D_S, \boldsymbol{q_S})\end{array}\right.$$

To simplify this problem, we can actually consider the same change of variables as in 7.17,

$$\left\{\begin{array}{l}\tilde q_i = (\alpha, R, T)^{-1} \cdot q_i = \alpha^{-1}R^\top q_i - \alpha^{-1}R^\top T \\ \tilde D = (\alpha, R, T)^{-1} \cdot D = \alpha^{-2}R^\top DR\end{array}\right.$$

which removes the action of the lie group $\alpha$-Isom$(\mathbb{R}^d)$ on the space of landmarks. The dynamic for these new variables is given by the infinitesimal action

$$\tilde\xi : \left\{\begin{array}{lll}\big(\alpha\text{-}\mathfrak{isom}(\mathbb{R}^d) \times C_0^k(\mathbb{R}^d, \mathbb{R}^d)\big) \times \mathcal{Q} & \to & T\mathcal{Q} \\ (s, A, \tau, u), (\alpha, R, T, \tilde D, \tilde q) & \mapsto & \big(s\alpha, AR, (s + A)T + \tau, 0, \alpha^{-1}R^\top u(\alpha R\tilde q + T)\big)\end{array}\right. \tag{8.4}$$

In particular, we now see that the metric $\tilde{D}$ becomes a constant of the dynamic. Moreover, by defining $\tilde{u} := \alpha^{-1} R^\top u(\alpha R \cdot + T)$, proposition 8.1 states that $|\tilde{u}|_{\tilde{D}} = |u|_{V_D}$ and we obtain the equivalent following matching problem (cf. proposition 7.19)

$$\inf_{(s_t, A_t, \tau_t, \tilde{u}_t) \in L^2([0,1], \alpha\text{-}\mathtt{isom}(\mathbb{R}^d) \oplus V_{D_S})} \tilde{J}(s_t, A_t, \tau_t, \tilde{u}_t) = \frac{1}{2} \int_0^1 |s_t|^2 + |A_t|^2 + |\tau_t|^2 + |\tilde{u}_t|^2_{V_{D_S}} \, dt$$
$$+ \mathcal{D}(\alpha_1, R_1, T_1, D_1, (\alpha_1, R_1, T_1) \cdot \tilde{\boldsymbol{q}}_1) \quad (8.5)$$

with dynamic given by

$$\begin{cases} (\dot{\alpha}_t, \dot{R}_t, \dot{T}_t) = (s_t \alpha_t, A_t R_t, (s_t + A_t) T_t + \tau_t) \\ \tilde{D}_t = D_S \\ \dot{\tilde{\boldsymbol{q}}}_t = \tilde{u}_t(\tilde{\boldsymbol{q}}_t) \\ (\alpha_0, R_0, T_0, D_0, \tilde{\boldsymbol{q}}) = (1, Id, 0, D_S, \boldsymbol{q_S}) \end{cases}$$

In the next proposition, we characterize the critical points of problem 8.5

**Proposition 8.3** (Hamiltonian characterization of 8.5)**.** *The hamiltonian $H : T^*\mathcal{Q} \to \mathbb{R}$ associated with the matching problem* (8.5) *is given by :*

$$H = \frac{1}{2} \left( |p^\alpha \alpha + p^\tau T^\top|^2 + |p^A R^\top + \mathrm{Sk}(p^\tau T^\top)|^2 + |p^\tau|^2 + \left| K_{\tilde{D}} \sum_i \delta_{\tilde{q}_i}^{\tilde{p}_i} \right|^2_{V_{\tilde{D}}} \right) \quad (8.6)$$

*where* $\mathrm{Sk}(A) = \frac{1}{2}(A - A^\top)$ *denotes the skew-symmetric of a matrix* $A \in M_d(\mathbb{R})$*.*

*Proof.* We define the pre-Hamiltonian corresponding to the problem (8.5)

$$H(\alpha, R, T, \tilde{D}, \tilde{\boldsymbol{q}}, p^\alpha, p^A, p^\tau, \tilde{p}^D, \tilde{\boldsymbol{p}}, s, A, \tau, \tilde{u}) = (p^\alpha \mid s\alpha) + (p^A \mid AR) + (p^\tau \mid \tau + (A+s)T) +$$
$$\sum_i (\tilde{p}_i \mid \tilde{u}(\tilde{q}_i)) - \frac{1}{2} \left( |s|^2 + |A|^2 + |\tau|^2 + |\tilde{u}|^2_{\tilde{D}} \right)$$

The condition on the controls

$$(\partial_s H, \partial_A H, \partial_\tau H, \partial_{\tilde{u}} H) = 0$$

therefore gives us

$$\begin{cases} s = p^\alpha \alpha + T^\top p^\tau \\ A = p^A R^\top + \mathrm{Sk}(p^\tau T^\top) \\ \tau = p^\tau \\ \tilde{u} = K_{\tilde{D}} \sum_i \delta_{\tilde{q}_i}^{\tilde{p}_i} \end{cases}$$

and the result follows. □

**Numerical result** We test our new model on the same source and target of Figure 8.1, using the same initial metric $D_\sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$, meaning we privilegize movements along the $x$-axis. However, note that in this new model, any rotation of the source induced by a rotation $R$ will also rotate the axis of the anisotropy (and we will get the new metric $RDR^\top$), meaning we cannot have the same issue as in Figure 8.2. Results are reported in Figure 8.3, where we cleary observe that no rotations are applied and the stretching is done along the $y$-axis even though the energy cost is higher.

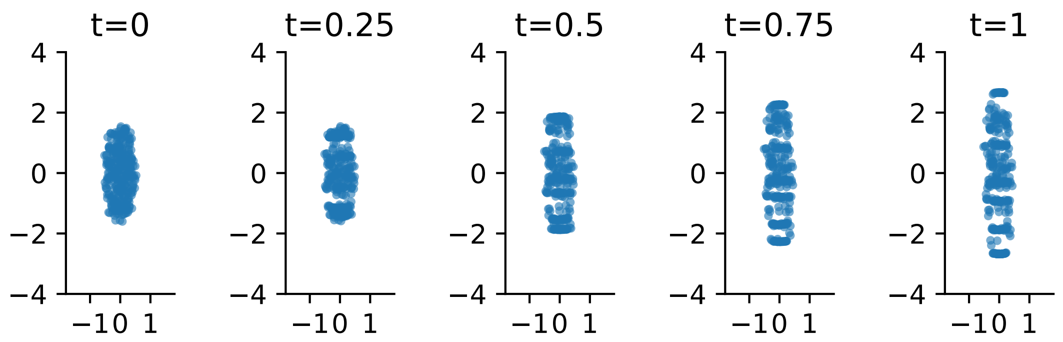


Figure 8.3: *Performing jointly rigid and non rigid registration minimizing problem 8.5, with an anisotropic kernel.* This experiment illustrates a matching task between the source (blue) and the target (red). The matching is performed through deformations generated by rotations and diffeomorphic deformation induced by a anisotropic gaussian kernel with scale $(\sigma_x, \sigma_y) = (1., 0.1)$.

#### 8.1.3.3 Invariance property of the non rigid part of the Hamiltonian

Following section 7.2.4, and by proposition 8.1 we see that the term in the Hamiltonian (8.6) corresponding to the non-rigid part

$$H(\tilde{D}, \tilde{\boldsymbol{q}}, \tilde{p}^D, \tilde{\boldsymbol{p}}) = \frac{1}{2} \sum_{i,j} \langle \tilde{p}_i, k_{\tilde{D}}(\tilde{q}_i, \tilde{q}_j)\tilde{p}_j \rangle \,.$$

is invariant by the action of the Lie group $\alpha$-Isom$(\mathbb{R}^d)$. This kernel is normalized in the sense that it allows to define Hamiltonians that are invariant under the action of rigid motions $\alpha$-Isom$(\mathbb{R}^d)$:

**Proposition 8.4** ($\alpha$-Isom$(\mathbb{R}^d)$-invariance of the Hamiltonian)**.** *Let* $\mathrm{Sk}(M) = \frac{1}{2}(M - M^\top)$ *denote the skew-symmetric of a matrix* $M \in \mathbb{R}^{d\times d}$*. The group* $\alpha$-Isom$(\mathbb{R}^d)$ *defines a canonical Hamiltonian action (cf. 3.1.3.1) on the symplectic manifold* $T^*(S_d^{++} \times \mathrm{Lmk}_n)$*, with momentum map*

$$\mu : \begin{array}{ccl} T^*(S_d^{++} \times \mathrm{Lmk}_n) & \longrightarrow & \alpha\text{-}\mathfrak{isom}(\mathbb{R}^d) \\ p^D, p & \longmapsto & \left(\sum_i \langle p_i, q_i\rangle + p^D D, \sum_i \mathrm{Sk}(p_i q_i^\top) + \mathrm{Sk}(p^D D), \sum_i p_i\right) \end{array} \tag{8.7}$$

*Moreover, the Hamiltonian* $H$ *defined previously is invariant by the action of* $\alpha$-Isom$(\mathbb{R}^d)$ *and the momentum map is a constant of the motion.*

*Proof.* The action of $\alpha$-Isom$(\mathbb{R}^d)$ on $T^*(S_d^{++} \times \mathrm{Lmk}_n))$ is given by

$$(\alpha, R, T) \cdot (D, q, p^D, p) = \left(\alpha^2 R D R^\top, \alpha R q + T, \alpha^{-2} R p^D R^\top, \alpha^{-1} R p\right)$$

which is the cotangent lift of the action of $\alpha$-Isom$(\mathbb{R}^d)$ on $\mathcal{Q}$, and therefore is a Hamiltonian action. The rest of the proof follows from proposition 8.1 and from Noether theorem. $\square$

#### 8.1.3.4 Transport of the metric by the diffeomorphisms

In the previous variational problem 8.3 (or equivalently 8.5), we transported the metric $D \in S_d^{++}$ representing the anisotropy of the shape by the action of the group $\alpha$-Isom$(\mathbb{R}^d)$.

A next step would be to define a transport of this metric by the group of diffeomorphisms too. We propose in this part a method to include transport of the anisotropy by the diffeomorphisms.

The tricky part is that there is no natural action of diffeomorphisms on the space $S_d^{++}$ of metrics in $\mathbb{R}^d$. However, even though there is no group actions of on this manifold, it is still possible to define an infinitesimal action of a the space of $C_0^k$-vector fields on $S_d^{++}$. In this sense, we define the infinitesimal transport of a metric $D$ by a vector field $u \in C_0^k(\mathbb{R}^d)^d$ as a vector bundle morphism

$$M: \begin{array}{ccc} (\mathbb{R}^d)^n \times C_0^k(\mathbb{R}^d, \mathbb{R}^d) & \longrightarrow & \mathbb{R}^{d\times d} \\ (q, u) & \longmapsto & M_q(u) \end{array}$$

**Remark 8.5.** *Here, the connection with the modular framework introduced in [46, 45, 47] becomes clearer. Indeed, we use the shape $q$ and the control $u$ to generate a matrix that transports the metric $D$ representing anisotropy in the shape.*

**Example 8.6** ($M_q(u) = \frac{1}{N}\sum_{i\leq N} du(q_i)$)**.** *Here the mapping $M : (\mathbb{R}^d)^n \times C_0^k(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R}^{d\times d}$ represents the mean of the jacobian on the shape $(q_i)_{i\leq N}$ and thus summarizes how the vector field $u$ acts on the anisotropy.*

Now the transport of a shape $(D, \boldsymbol{q}) \in S_d^{++} \times \mathrm{Lmk}_n(\mathbb{R}^d)$ by a vector field $u$ is given by

$$\xi^{\mathrm{Diff}}_{D,\boldsymbol{q}} = \left(M_{\boldsymbol{q}}(u)D + DM_{\boldsymbol{q}}(u)^\top, u(q_i)_i\right)$$

Now taking the sum of the two infinitesimal actions $\xi^{\alpha\text{-Isom}}$ and $\xi^{\mathrm{Diff}}$, we can thus modify the previous infinitesimal action and define

$$\xi : \begin{cases} \left(\alpha\text{-}\mathfrak{isom}(\mathbb{R}^d) \times C_0^k(\mathbb{R}^d, \mathbb{R}^d)\right) \times \mathcal{Q} & \to \quad T\mathcal{Q} \\ (s, A, \tau, u), (\alpha, R, T, D, q) & \mapsto \quad \big(s\alpha, AR, (s + A)T + \tau, 2sD + (A + M_q(u))D \\ & \qquad -D(A - M_q(u)^\top), sq + Aq + u(q) + sT + \tau\big) \end{cases}$$

The same change of variables

$$\begin{cases} \tilde{q}_i = (\alpha, R, T)^{-1} \cdot q_i = \alpha^{-1}R^\top q_i - \alpha^{-1}R^\top T \\ \tilde{D} = (\alpha, R, T)^{-1} \cdot D = \alpha^{-2}R^\top DR \\ \tilde{u} = \alpha^{-1}R^\top u(\alpha R \cdot +T) \end{cases}$$

gives a new dynamic associated with the infinitesimal action

$$\tilde{\xi} : \begin{cases} \left(\alpha\text{-}\mathfrak{isom}(\mathbb{R}^d) \times C_0^k(\mathbb{R}^d, \mathbb{R}^d)\right) \times \mathcal{Q} & \longrightarrow \quad T\mathcal{Q} \\ (s, A, \tau, \tilde{u}), (\alpha, R, T, \tilde{D}, \tilde{q}) & \longmapsto \quad \big(s\alpha, AR, (s + A)T + \tau, \tilde{M}_{\alpha,R,T,\tilde{q}}(\tilde{u})\tilde{D} + \\ & \qquad \tilde{D}\tilde{M}_{\alpha,R,T,\tilde{q}}(\tilde{u})^\top,\ \tilde{u}(\tilde{q})\big) \end{cases}$$

where $\tilde{M} : \mathcal{Q} \times C_0^k(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R}^{d\times d}$ is given by

$$\tilde{M}_{\alpha,R,T,\tilde{q}}(\tilde{u}) = R^\top M_{\alpha R\tilde{q}+T}\left(d\varphi_{(\alpha,R,T)}\tilde{u} \circ \varphi_{(\alpha,R,T)^{-1}}\right) R\,.$$

In such case, the new variable $\tilde{D}$ is not anymore a constant of the dynamic (as in (8.4)) since it is still transported by the vector field $u$. The matching problem associated with this framework leads to the minimization of the following new functional

$$\begin{aligned} \inf_{(s,A,\tau,u)\in \mathrm{Hor}_{L_2}(\overline{V})} \tilde{J}(s, A, \tau, \tilde{u}) = \frac{1}{2}\int_0^1 |s|^2 + |A|^2 + |\tau|^2 + |\tilde{u}|^2_{V_{\tilde{D}}}\, dt \\ +\mathcal{D}(\alpha_1, R_1, T_1, D_1, (\alpha_1, R_1, T_1) \cdot \tilde{\boldsymbol{q}}_1) \end{aligned} \tag{8.8}$$

where $(\tilde{D}_t, R_t, T_t, \tilde{q}_t)$ satisfies the dynamic

$$\begin{cases} (\dot{\alpha}_t, \dot{R}_t, \dot{T}_t, \dot{\tilde{D}}_t, \dot{\tilde{\boldsymbol{q}}}_t) = \tilde{\xi}_{(\tilde{D}_t, R_t, T_t, \tilde{\boldsymbol{q}}_t)}(s_t, A_t, \tau_t, u_t), \\ (\alpha_0, R_0, T_0, \tilde{D}_0, \tilde{q}_0) = (1, I_d, 0, D_S, \boldsymbol{q_S}) \end{cases}$$

In the next proposition, we compute the Hamiltonian associated with the variational problem (8.8):

**Proposition 8.7.** *The hamiltonian $H : T^*\mathcal{Q} \to \mathbb{R}$ associated with the matching problem* (8.8) *is given by :*

$$H = \frac{1}{2}(|p^\alpha \alpha + p^\tau T^\top|^2 + |p^A R^\top + \mathrm{Sk}(T^\top p^\tau)|^2 + |p^\tau|^2) + \frac{1}{2}\left| K_{\tilde{D}}\left(\sum_i \delta_{\tilde{q}_i}^{\tilde{p}_i} + 2M^*_{\alpha,R,T,\tilde{\boldsymbol{q}}}(\tilde{p}^D \tilde{D})\right)\right|^2_{V_{\tilde{D}}} \tag{8.9}$$

*Proof.* We define the pre-Hamiltonian corresponding to the problem (8.8)

$$H(\alpha, R, T, \tilde{D}, \tilde{\boldsymbol{q}}, p^\alpha, p^A, p^\tau, \tilde{p}^D, \tilde{\boldsymbol{p}}, s, A, \tau, \tilde{u}) = (p^\alpha \mid s\alpha) + (p^A \mid AR) + (p^\tau \mid \tau + (A+s)T) + (\tilde{p}^D \mid M_{\alpha,R,T,\tilde{q}}(\tilde{u})\tilde{D} + \tilde{D}M_{\alpha,R,T,\tilde{q}}(\tilde{u})^\top) + \sum_i (\tilde{p}_i \mid \tilde{u}(\tilde{q}_i)) - \frac{1}{2}\left(|s|^2 + |A|^2 + |\tau|^2 + |\tilde{u}|^2_{\tilde{D}}\right)$$

The condition on the controls

$$(\partial_s H, \partial_A H, \partial_\tau H, \partial_{\tilde{u}} H) = 0$$

therefore gives us

$$\begin{cases} s = p^\alpha \alpha + T^\top p^\tau \\ A = p^A R^\top + \mathrm{Sk}(p^\tau T^\top) \\ \tau = p^\tau \\ \tilde{u} = K_{\tilde{D}}\left(\sum_i \delta_{\tilde{q}_i}^{\tilde{p}_i} + 2M^*_{\alpha,R,T,\tilde{\boldsymbol{q}}}(p^D \tilde{D})\right) \end{cases}$$

and the result follows. □

## 8.2 The special case of images, and anisotropic scaling

In computational anatomy, we are particularly interested in the matching of a template image $I_S \in \mathcal{I}$ onto a target image $I_T \in \mathcal{I}$ by a smooth deformation that produces a one-to-one correspondence between pixels/voxels for applications in medical imaging. This section presents how the large deformation model can be enriched with isometries following the previous setting, and anisotropic scalings and their actions on images. This particular example actually does not really fit in the general framework described before in chapter 6 since the action of diffeomorphisms on images is only continous, and the action of isometries and scaling is ill-defined. We can still recover some of the results proved before and keep track of the regularity of images during the transport by diffeomorphisms.

Moreover, we will also suppose, following section 8.1, that the finite dimensional groups of scalings and rotations also act on the metric of the kernel generating the vector fields for the non rigid part, allowing to follow the anisotropy of the shape. The main difference with the previous section is that we introduce now an anisotropic scaling, meaning we will not apply the same scaling on each axis.

### 8.2.1 Action on images

#### 8.2.1.1 Transport by diffeomorphisms

Let $\mathcal{I} := L^2(\Omega, \mathbb{R})$ be the set of grey scale images on the image domain $\Omega \subset \mathbb{R}^d$ and we consider a source target $I_S \in \mathcal{I}$ that we want to match with a target $I_T \in \mathcal{I}$. We recall that in the large deformation framework, the trajectory of an image is defined as the action of a flow of diffeomorphism in $\mathrm{Diff}_{C_0^k}(\Omega)$ on the template image $I_t = \varphi_t \cdot I_S := I_S \circ \varphi_t^{-1}$. This action is only continuous because of the composition on the right by the inverse of $\varphi$. This means, in particular, that we need more regularity on the image $I$ to define the infinitesimal action. In particular, if $I \in H^1(\Omega, \mathbb{R})$, we can differentiate the action and define

$$\xi : \begin{array}{ccc} C_0^k(\Omega, \mathbb{R}^d) & \longrightarrow & \mathcal{I} \\ v & \longmapsto & -\langle \nabla I, v \rangle \end{array}$$

Its evolution can be derived from the dynamic of the diffeomorphism :

$$\dot{I}_t = v_t \cdot I_t := -\langle \nabla I_t, v_t \rangle, \quad I_0 = I_S \tag{8.10}$$

where $v_t = \dot{\varphi}_t \circ \varphi_t^{-1}$ is the Eulerian derivative.

**Proposition 8.8** (Flow of images)**.** *Suppose $I_S \in H^1(\Omega, \mathbb{R})$ and $v \in L^2([0,1], C_0^1(\Omega, \mathbb{R}^d))$. Then there exists a unique global solution of the evolution equation* (8.10)

$$\dot{I}_t = -\langle \nabla I_t, v_t \rangle, \quad I_0 = I_S \tag{8.11}$$

*Moreover this unique solution $I_t$ also satisfies the integrated equation*

$$I_t = I_S \circ \varphi_t^{-1} \tag{8.12}$$

*where $\varphi_t$ is the flow of $v_t$.*

**Remark 8.9.** *The integrated formulation implies, in particular that the flow $I_t$ will belong to $H^1(\Omega, \mathbb{R})$ for all time $t$, but is only absolutely continuous when $I_t$ lies in $L^2$.*

*Proof.* Let $v_t \in L^2([0,1], C_0^1(\Omega, \mathbb{R}^d))$. We introduce the curve $\varphi_t \in AC_{L^2}([0,1], \mathrm{Diff}_{C_0^1}(\mathbb{R}^d))$ flow of $v_t$, i.e. solution of the equation

$$\dot{\varphi}_t = v_t \circ \varphi_t, \quad \varphi_0 = \mathrm{id}$$

Since the restriction of the action

$$\begin{array}{ccc} \mathrm{Diff}_{C_0^1}(\mathbb{R}^d) \times H^1(\Omega, \mathbb{R}) & \longrightarrow & L^2(\Omega, \mathbb{R}) \\ \varphi, I & \longmapsto & I \circ \varphi^{-1} \end{array}$$

is $C^1$, the curve $t \mapsto I_S \circ \varphi_t^{-1} \in H^1(\Omega, \mathbb{R})$ is therefore absolutely continuous in $\mathcal{I} = L^2(\Omega, \mathbb{R})$ and is solution of (8.11).

Conversely, suppose $t \mapsto I(t)$ is an absolutely continuous curve in $\mathcal{I}$, such that for any $t \in I$, $I_t \in H^1(\Omega, \mathbb{R})$ and $I_t$ is solution of (8.11). We prove now that for all $t \in [0,1]$,

$$I_t \circ \varphi_t = I_S.$$

The curve $t \mapsto I_t \circ \varphi_t$ is not necessarily absolutely continuous in $\mathcal{I}$, since we only have $I_t \in \mathrm{AC}_{L^2}([0,1], \mathcal{I})$. However for any $h \in C_0^\infty(\Omega, \mathbb{R})$, we can derivate $t \mapsto \int_\Omega \langle I_t \circ \varphi_t(x), h(x) \rangle dx$

and we get

$$\frac{d}{dt}\int_\Omega I_t \circ \varphi_t(x)\, h(x)dx = -\int_\Omega \langle \nabla I_t \circ \varphi_t(x), v_t(x)\rangle h(x)dx + \int_\Omega dI_t\left(\varphi_t(x)\right) v_t \circ \varphi_t h(x)dx$$
$$= 0$$

from which the result follows. □

We next extend this framework using the action of scalings and isometries.

#### 8.2.1.2 Anisotropic scalings and isometries

We consider here the group of anisotropic scalings and isometries. We define an anisotropic scaling as an element $\alpha \in \mathbb{R}^d$ that acts on $\mathbb{R}^d$ by

$$\alpha \cdot x = \mathsf{D}_\alpha x = (\alpha_1 x_1, .., \alpha_d x_d)$$

where $\mathsf{D}_\alpha$ is the diagonal matrix with $\alpha_i$ as the elements on the diagonal. We then define the direct product group

$$\mathsf{D}_\alpha\text{-Isom}(\mathbb{R}^d) := \mathbb{R}^d_{>0} \times SO_d \times \mathbb{R}^d$$

with group law given by

$$\begin{cases} (\alpha, R, T)\cdot(\alpha', R', T') = (\alpha\alpha', RR', T+T') \\ (\alpha, R, T)^{-1} = (\alpha^{-1}, R^\top, -T) \end{cases}$$

Note that here that the finite dimensional Lie group is a direct product of scalings, rotations and translations, and not the semidirect product from 8.1.2.1. Indeed, the action we define from this group on images will also depend on extra-information we add to this image, in order to rotate around the center and scale along particular axes. We therefore suppose any image $I$ is associated with an intrinsic orthogonal basis, encoded by a rotation $R_I \in \mathrm{SO}_d$, and an origin of the scaling $O_I$. This adds intrinsic natural information of the images and provides new coordinates to perform the rigid and scaling deformations. The anisotropic scaling will act and stretch each axis of this basis, and the rotation will be centered around $O_I$. We therefore define the action of $(\alpha, R, T)$ on the element $(I, R_I, O_I)$, by

$$(\alpha, R, T)\cdot(I, R_I, O_I) = (I', RR_I, O_I + T) \tag{8.13}$$

where the image $I'$ is given by

$$I'(x) = \begin{cases} I\left(R_I \mathsf{D}_{\alpha^{-1}} R_I^\top R^\top (x - O_I - T) + O_I\right) & \text{if } R_I \mathsf{D}_{\alpha^{-1}} R_I^\top R^\top (x - O_I - T) + O_I \in \Omega \\ 0 & \text{else} \end{cases}$$

Note that this does not really define an action since the images are defined on the open set $\Omega \subset \mathbb{R}^d$ that is not necessarily stable by the action of isometries and scaling. This means that a loss or a lack of information may occur when the rigid part is too large. However, in some applications, the support of images is strictly contained within $\Omega$ and we can rigidly transport them without loss of information.

**Proposition 8.10** (Action property)**.** *Let $(I, R_I, O_I) \in \mathcal{I} \times SO_d \times \mathbb{R}^d$, and denote by* $\mathrm{Supp}(I) \subset \Omega$ *its support. Let $(\alpha, R, T), (\alpha', R', T') \in \mathsf{D}_\alpha\text{-Isom}(\mathbb{R}^d)$, and we define*

$$\begin{aligned} S_1 &= R'R_I\mathsf{D}_\alpha R_I^\top(\mathrm{Supp}(I) - O_I) + O_I + T' \\ S_2 &= RR'R_I\mathsf{D}_{\alpha'} R_I^\top R'^\top(S_1 - O_I - T') + O_I + T' + T \end{aligned}$$

*We suppose morever that*

$$S_1, S_2 \subset \Omega \tag{8.14}$$

*Then we get that*

$$(\alpha, R, T) \cdot ((\alpha', R', T') \cdot (I, R_I, O_I)) = (\alpha\alpha', RR', T' + T) \cdot (I, R_I, O_I) \tag{8.15}$$

*and the set $S_2$ is exactly the support of the transported image*

*Proof.* The condition (8.14) is exactly the necessary condition to get a complete rigid transport of the image $I$ without any loss of information. We denote $I' \in \mathcal{I}$ the image such that

$$(\alpha', R', T') \cdot (I, R_I, O_I) = (I', R'R_I, O_I + T').$$

and $I'' \in \mathcal{I}$ such that

$$(\alpha, R, T) \cdot (I', R'R_I, O_I + T') = (I'', RR'R_I, O_I + T' + T).$$

Let $z \in \Omega$, and $x, y \in \mathbb{R}^d$ such that

$$\begin{cases} z = RR'R_I\mathsf{D}_{\alpha'}R_I^\top R'^\top(y - O_I - T') + O_I + T' + T \\ y = R'R_I\mathsf{D}_\alpha R_I^\top(x - O_I) + O_I + T'. \end{cases}$$

In particular we get that $x \in \mathrm{Supp}(I)$ if and only if $y \in S_1$, and by hypothesis (8.14),

$$I'(y) = I(x)$$

which also means that $S_1 = \mathrm{Supp}(I')$. Similarly, $z \in S_2$ if and only if $y \in S_1 = \mathrm{Supp}(I')$, if and only if $x \in \mathrm{Supp}(I)$, and therefore

$$I''(z) = I'(y) = I(x)$$

This also means in particular that $S_2 = \mathrm{Supp}(\mathrm{I}'')$. Finally, we have

$$\begin{aligned} z &= RR'R_I\mathsf{D}_{\alpha'}R_I^\top R'^\top(y - O_I - T') + O_I + T' + T \\ &= RR'R_I\mathsf{D}_\alpha R_I^\top R'^\top\left(\left(R'R_I\mathsf{D}_{\alpha'}R_I^\top(x - O_I) + O_I + T'\right) - O_I - T'\right) + O_I + T' + T \\ &= RR'R_I\mathsf{D}_\alpha R_I^\top R'^\top\left(R'R_I\mathsf{D}_{\alpha'}R_I^\top(x - O_I)\right) + O_I + T' + T \\ &= RR'R_I\mathsf{D}_{\alpha\alpha'}R_I^\top(x - O_I) + O_I + T' + T. \end{aligned}$$

so that we get

$$(\alpha\alpha', RR', T' + T) \cdot (I, R_I, O_I) = (I'', RR'R_I, O_I + T' + T).$$

□

#### 8.2.1.3 The coupled action

In this case, both the actions on the space of images that we have defined are not compatible in the sense of 7.2.2, so we cannot define the action of a semidirect product group on $\mathcal{I}$ as done before. However, we can directly consider the second formulation (after the change of variable) and define the product group:

$$\mathsf{D}_\alpha\text{-Isom}(\mathbb{R}^d) \times \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$$

with group law

$$\begin{cases} (\alpha, R, T, \varphi) \cdot (\alpha', R', T', \varphi') = (\alpha\alpha', RR', T' + T, \varphi \circ \varphi') \\ (\alpha, R, T, \varphi)^{-1} = (\alpha^{-1}, R^\top, -T, \varphi). \end{cases}$$

This defines a Banach half-Lie group as a direct product of half-Lie groups that acts on the space of shapes $\mathcal{I} \times SO_d \times \mathbb{R}^d$ by

$$(\alpha, R, T, \varphi) \cdot (R_I, O_I, I) = (I \circ \varphi^{-1}, RR_I, O_I + T). \tag{8.16}$$

The interactions between the actions of the rigid part and the diffeomorphic part will take place in the data attachment term, as detailed in the next section.

### 8.2.2 Matching problem and Hamiltonian equations

Considering the action (8.13), we can define a matching problem between a source image $I_S$ and a target $I_T$, that takes into account both the actions of the rigid parts and the diffeomorphisms.

#### 8.2.2.1 Data attachment term

We define a dissimiliarity term with the target image $I_T$ using the action of $\mathsf{D}_\alpha\text{-Isom}(\mathbb{R}^d)$

$$\mathcal{D}(\alpha, R, T, I) = \gamma \|(\alpha, R, T) \cdot I_S - I_T\|_{L^2}^2 + (1 - \gamma)\|(\alpha, R, T) \cdot I - I_T\|_{L^2}^2$$

with $0 < \gamma < 1$, and where we suppose we use the action (8.13) of $(\alpha, R, T)$ on the augmented images $(I_S, R_{I_S}, O_{I_S})$ and $(I, R_{I_S}, O_{I_S})$. This allows to relax the ending condition. Moreover we use here two terms, the first one allows to evaluate the rigid and scaling alignment only, while the second term corresponds to the cost for the total deformation. In particular, the variable $I$ will contain the diffeomorphic deformation.

#### 8.2.2.2 Matching problem

The matching problem associated with this framework is

$$\inf_{(s,A,\tau,v)} J(s, A, \tau, v) = \frac{1}{2} \int_0^1 |s_t|^2 + |A_t|^2 + |\tau_t|^2 + |v_t|^2 \, dt + \mathcal{D}(\alpha_1, R_1, T_1, I_1) \tag{8.17}$$

$$\text{s.t} \quad \begin{cases} \dot{\alpha}_t = s_t \alpha_t \\ \dot{R}_t = A_t R_t \\ \dot{T}_t = \tau_t \\ \dot{I}_t = -\langle \nabla I_t, v_t \rangle \end{cases}$$

#### 8.2.2.3 Hamiltonian equations

As before, we can associate to this matching problem a pre-Hamiltonian to compute the critical points

$$H(\alpha, R, T, I, p^\alpha, p^A, p^\tau, p^I, s, A, \tau, v) = \\ (p^\alpha \mid \alpha) + (p^A \mid AR) + (p^\tau \mid \tau) + \big(p^I \mid \langle -\nabla I, v\rangle\big) - \frac{1}{2}\left(|s|^2 + |A|^2 + |\tau|^2 + |v|_V^2\right)$$

so that the critical points of the energy satisfy the Hamiltonian equations. The conditions on the control $(\partial_s H, \partial_A H, \partial_\tau H, \partial_v H) = 0$ leads to

$$\begin{cases} s = p^s\alpha \\ A = p^A R^\top \\ \tau = p^\tau \\ v = -K_V(p^I \nabla I) \end{cases} \tag{8.18}$$

and the Hamiltonian equations simplify to

$$\begin{cases} \dot{\alpha}_t = s_t\alpha_t \\ \dot{R}_t = A_t R_t \\ \dot{T}_t = \tau_t \\ \dot{I}_t(x) = -\langle \nabla I_t(x), v_t(x)\rangle \end{cases}, \qquad \begin{cases} \dot{p}^s_t = -s_t p^s \\ \dot{p}^A_t = A_t p^A_t \\ \dot{p}^T_t = 0 \\ \dot{p}^I_t = -\nabla\cdot(v_t p^I_t) \end{cases} \tag{8.19}$$

with initial conditions $(\alpha_0, R_0, T_0, I_0) = (1_d, \mathrm{id}, 0, I_S)$.

#### 8.2.2.4 Constraints

Once again, if the Hilbert space $V$ generating the vector fields is invariant by the action of rotations and translations (for example with a Gaussian RKHS), there are some symmetries in the Hamiltonian. We will not be able to apply directly Noether's theorem and symplectic reduction since the transport of images by rotations and translations is not exactly an action unless the support is small enough, as explained previously. However, we can still define constraints on covectors, as done in section 7.2.4

$$\begin{cases} \int_\Omega p^I(x)(\nabla I(x)x^\top - x\nabla I(x)^\top)dx = 0 \\ \int_\Omega p^I(x)\nabla I(x)dx = 0 \end{cases} \tag{$C_I$}$$

The first constraint corresponds to the angular momentum and the second one to the translation momentum. We will prove by hand, under some assumptions that we will precise, a variant of Noether's theorem adapted to our setting.

**Proposition 8.11** (Constraints conservation). *Let* $\big((\alpha_t, R_t, T_t, I_t), (p^\alpha_t, p^A_t, p^\tau_t, p^I_t)\big)$ *satisfy the Hamiltonian equations* (8.19). *Suppose also that* $\mathrm{Supp}(I_t) \subset \Omega$ *for all* $t$*, then the constraints* $(C_I)$ *are conserved during the dynamic.*

*Proof.* Let introduce the flow $\varphi_t$ of $v_t$, i.e. satisfying $\dot{\varphi}_t = v_t \circ \varphi_t$, with initial condition $\varphi_0 = \mathrm{id}$. Since $I_S \in H^1(\Omega, \mathbb{R})$, and as stated in proposition 8.8, we get that

$$I_t = I_S \circ \varphi_t^{-1}.$$

Similarly the covector $p_t^I$ satisfies the integrated formulation

$$p_t^I = p_0^I \circ \varphi_t^{-1} |d\varphi_t^{-1}|$$

which corresponds to the cotangent lift of the action of $\varphi_t$ on $p_0^I$. We can now prove that the first constraint (corresponding to the angular momentum) is conserved. Recall the notation of the skew-symmetric mapping $\mathrm{Sk} : \mathbb{R}^{d\times d} \to \mathfrak{so}_d$, $\mathrm{Sk}(M) = \frac{1}{2}(M - M^\top)$ so that

$$\frac{1}{2}\int_\Omega p_t^I(x)(\nabla I_t(x)x^\top - x\nabla I_t(x)^\top)dx = \mathrm{Sk}\left(\int_\Omega p_t^I(x)\nabla I_t(x)x^\top dx\right)$$

Moreover, using the change of variable induced by $\varphi_t^{-1}$, we see that

$$\mathrm{Sk}\left(\int_\Omega p_t^I(x)\nabla I_t(x)x^\top dx\right) = \mathrm{Sk}\left(\int_\Omega p_0^I(x)\, d\varphi_t(x)^{-\top}\nabla I_S(x)\varphi_t(x)^\top dx\right)$$

This equality allows to reinterpret the constraint as the angular momenta of the particles $\varphi_t(x)$ with covectors $p_0^I(x)\, d\varphi_t(x)^{-\top}\nabla I_S(x)$. By derivating this expression, we then get

$$\begin{aligned}
\frac{d}{dt}\mathrm{Sk}\big(\int_\Omega p_t^I(x)\nabla I_t(x)x^\top dx\big) &= \frac{d}{dt}\mathrm{Sk}\left(\int_\Omega p_0^I(x)\, d\varphi_t(x)^{-\top}\nabla I_S(x)\varphi_t(x)^\top dx\right)\\
&= \mathrm{Sk}\left(-\int_\Omega p_0^I(x)\, dv_t(\varphi_t(x))^\top d\varphi_t(x)^{-\top}\nabla I_S(x)\varphi_t(x)^\top dx\right)\\
&+ \mathrm{Sk}\left(\int_\Omega p_0^I(x)\, d\varphi_t(x)^{-\top}\nabla I_S(x)\left(v_t\circ\varphi_t(x)\right)^\top dx\right)
\end{aligned}$$

We recall that the condition on the controls (8.18) give that

$$v_t(x) = -\int_\Omega k_\sigma(x,y)p_t^I(y)\nabla I_t(y)dy = -\int_\Omega k_\sigma(x,\varphi_t(y))p_0^I(y)d\varphi_t(y)^{-\top}\nabla I_S(y)dy$$

so that the first term of the previous integral becomes

$$\begin{aligned}
&\mathrm{Sk}\left(-\int_\Omega p_0^I(x)\, dv_t(\varphi_t(x))^\top d\varphi_t(x)^{-\top}\nabla I_S(x)\varphi_t(x)^\top dx\right)\\
&= \mathrm{Sk}\left(\int_\Omega\int_\Omega p_0^I(x)p_0^I(y)\nabla_1 k_\sigma\left(\varphi_t(x),\varphi_t(y)\right)\left\langle d\varphi_t(y)^{-\top}\nabla I_S(y), d\varphi_t(x)^{-\top}\nabla I_S(x)\right\rangle\varphi_t(x)^\top dx\right)\\
&= \mathrm{Sk}\left(\int_\Omega\int_\Omega\langle\tilde{p}_t(x),\tilde{p}_t(y)\rangle\nabla_1 k_\sigma\left(\varphi_t(x),\varphi_t(y)\right)\varphi_t(x)^\top dx\right)
\end{aligned}$$

where $\tilde{p}_t(x) = p_0^I(x)d\varphi_t(x)^{-\top}\nabla I_S(x)$. Since the kernel $k_\sigma$ is invariant by rotation, this means in particular that

$$\nabla_1 k_\sigma\left(\varphi_t(x),\varphi_t(y)\right)\varphi_t(x)^\top = -\nabla_1 k_\sigma\left(\varphi_t(y),\varphi_t(x)\right)\varphi_t(x)^\top$$

and we get then that the first term $\mathrm{Sk}\left(\int_\Omega p_0^I(x)\, dv_t(\varphi_t(x))^\top d\varphi_t(x)^{-\top}\nabla I_S(x)\varphi_t(x)^\top dx\right)$ is equal to 0. Similarly for the second term, we have

$$\begin{aligned}
&\mathrm{Sk}\left(\int_\Omega p_0^I(x)\, d\varphi_t(x)^{-\top}\nabla I_S(x)\left(v_t\circ\varphi_t(x)\right)^\top dx\right)\\
&= -\mathrm{Sk}\left(\int_\Omega\int_\Omega p_0^I(x)\, d\varphi_t(x)^{-\top}\nabla I_S(x)k_\sigma\left(\varphi_t(x),\varphi_t(y)\right)p_0^I(y)\left(d\varphi_t(y)^{-\top}\nabla I_S(y)\right)^\top dx\right)\\
&= -\mathrm{Sk}\left(\int_\Omega\int_\Omega k_\sigma\left(\varphi_t(x),\varphi_t(y)\right)\tilde{p}_t(x)\tilde{p}_t(y)^\top dx\right)\\
&= 0.
\end{aligned}$$

Therefore we finally get that the angular momentum is conserved, i.e.

$$\frac{d}{dt}\,\mathrm{Sk}\left(\int_\Omega p_t^I(x)\nabla I_t(x)x^\top dx\right) = 0$$

The conservation of the second constraint, corresponding to the translation momentum can be proved the same way using the change of variable induced by $\varphi_t$. It is actually easier to prove it using Stokes theorem directly :

$$\begin{aligned}
\frac{d}{dt}\int_\Omega p_t^I(x)\nabla I_t(x)dx &= -\int_\Omega \nabla\cdot(v_t p_t^I)(x)\nabla I_t(x)dx - \int_\Omega p_t^I(x)\nabla\langle\nabla I_t(x), v_t(x)\rangle dx \\
&= -\int_\Omega \nabla\cdot(v_t p_t^I)(x)\nabla I_t(x)dx + \int_\Omega \nabla p_t^I(x)\langle\nabla I_t(x), v_t(x)\rangle dx \\
&= -\int_\Omega \big((\nabla\cdot v_t(x))p_t^I(x) + \langle v_t(x), \nabla p_t^I(x)\rangle\big)\nabla I_t(x)dx \\
&+ \int_\Omega \nabla p_t^I(x)\langle\nabla I_t(x), v_t(x)\rangle dx \\
&= \int_\Omega\int_\Omega \nabla\cdot\big(k_\sigma(.,y)p_t^I(y)\nabla I_t(y)\big)(x)p_t^I(x)\nabla I_t(x)dxdy \\
&= \int_\Omega\int_\Omega \nabla_1 k_\sigma(x,y)p_t^I(y)\nabla I_t(y)p_t^I(x)\nabla I_t(x)dxdy \\
&= 0,
\end{aligned}$$

which completes the proof. □

### 8.2.3 Algorithm and numerical examples

In this section, we present an algorithmic procedure to compute the coupled rigid and non-rigid matching between a source and a target image, and thereby minimize problem (8.17). Building on this model, we conduct a series of numerical experiments using the `Demeter` library[1] [36].

#### 8.2.3.1 Algorithmic procedure

It should be noted that in solving a rigid and scaling registration, the main difficulty lies in avoiding convergence to a local minimum, particularly for images. In the examples below, we will carry out a preliminary exploration of the rigid registrations to identify plausible solutions. Then we will start with the found rotation, translation and scaling the joint minimization for the total group of deformations, including the non rigid part and the rigid part. This whole procedure gives the following algorithm :

[1] `https://github.com/antonfrancois/Demeter_metamorphosis`

**Algorithm 1:** Rigid and diffeomorphic registrations between two images

**Input:** $I_S$, $I_T$ : source and target images to register

**Output:** Estimated matching between $I_S$ and $I_T$

**Step 1 – Barycentric centering**

Compute the barycenter $b_S$ of $I_S$;

Compute the barycenter $b_T$ of $I_T$;

Define centered source and target $I_{S,b}(x) = I_S(x + b_S)$ and $I_{T,b}(x) = I_T(x + b_T)$;

**Step 2 – Finding candidates for rotation**

**for** *every rotation $r_i$ in a list of candidate rotations* **do**

  Apply a simple rigid geodesic shooting from $r_i$;

  Compute the data attachment term: $\|I_{S,b} \circ R_i^\top - I_{T,b}\|_2^2$;

**Step 3 – Scaling and rigid optimization**

Select the $N$ best rotations $r_i$ using the previous criterion;

Optimize $\alpha$, $R$ and $T$ from this $N$ candidates;

**Step 4 – Rigid and Diffeomorphic optimization**

Start an optimization and minimize (8.17) to find the best rigid and diffeomorphic matching from the initial conditions obtained in step 3;

**Step 5 – Compute final translation**

Apply the final translation to result obtained : $b_S + T - b_T$;

**return** *Complete matching between $I_S$ and $I_T$*

#### 8.2.3.2 Demonstration on toy-example

Figures 8.4 and 8.5 illustrate two registrations on a toy example where the goal is to align two stars. In this example, one star has to be matched with one displaced and rotated on one side, while on the other side the branches of the star vary in length and the central white ring has undergone a deformation. Both approaches succeed in producing a final image identical to the target (see (a), figures 8.4 and 8.5), but they yield different rigid deformations. It should be emphasized that neither registration is intrinsically better than the other. In a real scenario, one or the other solution might be preferable, and this choice should be guided by introducing priors into the registration, for instance through data attachment.

In Figure 8.4, the rigid and scaling registration obtained is optimal with respect to the shape of the star. It is the best according to most data attachment criteria, since the gray surface is larger than the white one. From the diffeomorphic perspective, this rigid transformation requires the least amount of global deformation. However, to compensate, the ring causes the handle to disappear and then reappear.

In Figure 8.5, the affine transformation successfully aligns the white ring with the handle (panels b and d). Qualitatively, this seems to correspond to a better rigid registration. Quantitatively, however, it performs less well with respect to the $L^2$ data attachment term (also called SSD.

In Figure 8.6, we present the result of LDDMM following a rigid registration, similar to that in Figure 8.4, and comparable to what would be obtained using SSD as the data attachment term. The scale parameters, the number of integration steps and the scales chosen in the kernel are kept identical. It can be observed that here, the deformation obtained with LDDMM is insufficient to achieve proper alignment. Obviously, we could

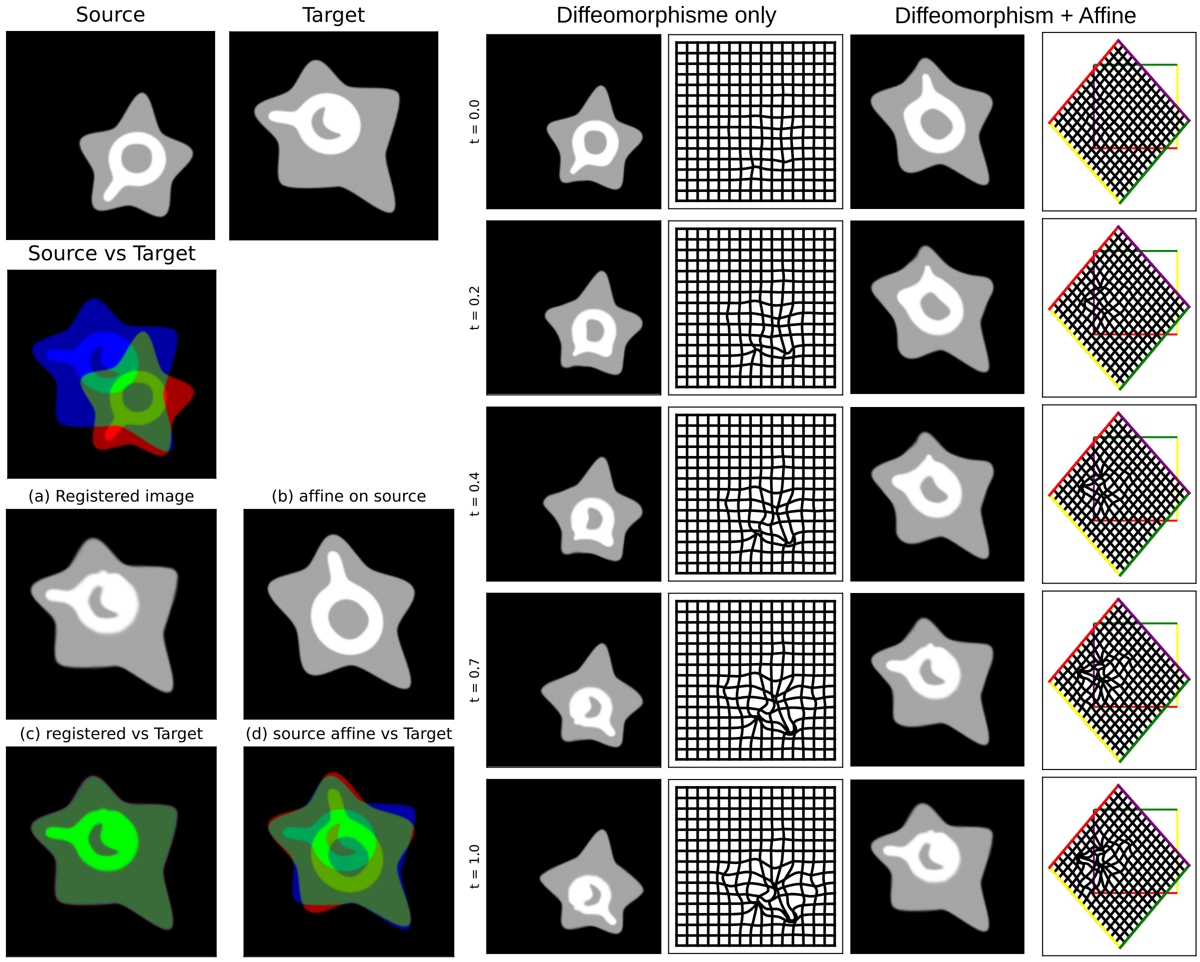


Figure 8.4: **Toy example : matching of star-like images by optimizing problem** (8.17)**, using algorithm 1. (top-left)** Source and target images, along with their superimposition. Green areas indicate regions where pixel intensities are similar, while blue and red correspond to differences from one image or the other. **(bottom-left)** Final registration: total deformation in (a) and (c), and final rigid deformation in (b) and (d). **(right)** Temporal integration with visualization of the deformations. The two left columns represent the evolution of $I_t$ in problem (8.17), together with the diffeomorphism applied to a grid. The third column represents the evolution of $(\alpha_1, R_1, T_1) \cdot I_t$, in order to better visualize the comparison with the target. On the right, in the grids subjected to an affine transformation, the colored borders help to visualize the orientation.

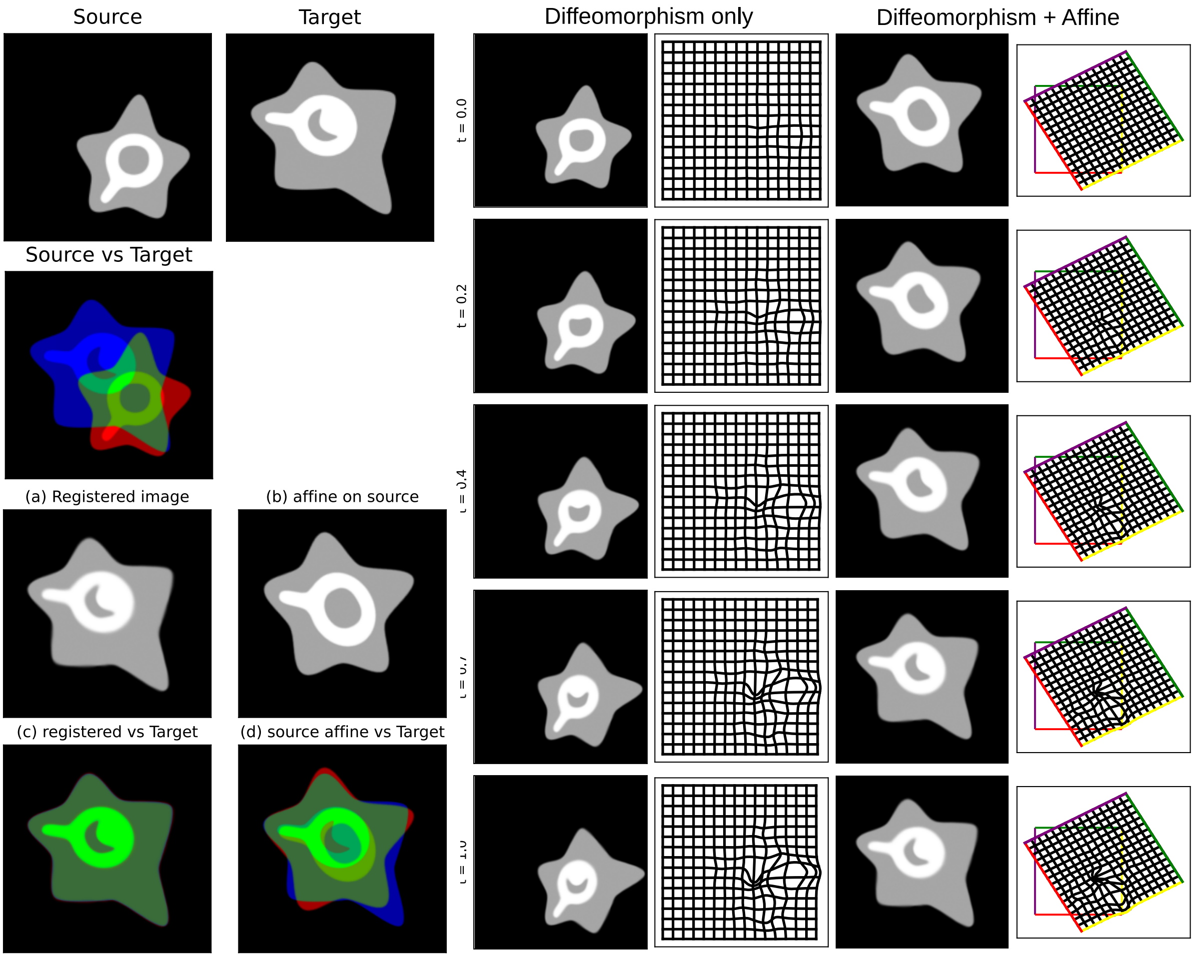


Figure 8.5: **Toy example : matching of star-like images by optimizing problem** (8.17)**, using algorithm 1. (top-left)** Source and target images, along with their superimposition. Green areas indicate regions where pixel intensities are similar, while blue and red correspond to differences from one image or the other. **(bottom-left)** Final registration: total deformation in (a) and (c), and final rigid deformation in (b) and (d). **(right)** Temporal integration with visualization of the deformations. The two left columns represent the evolution of $I_t$ in problem (8.17), together with the diffeomorphism applied to a grid. The third column represents the evolution of $(\alpha_1, R_1, T_1) \cdot I_t$, in order to better visualize the comparison with the target. On the right, in the grids subjected to an affine transformation, the colored borders help to visualize the orientation.

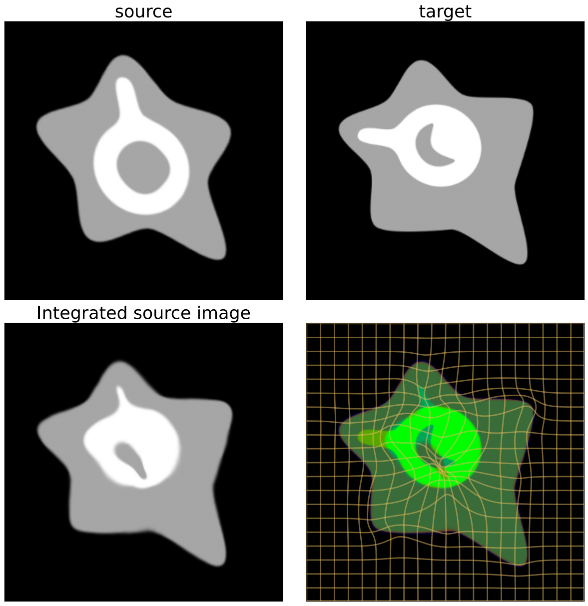


Figure 8.6: classical LDDMM failed registration on the toy-example from Fig. 8.4 rigidly aligned.

obtain better results in this 2-steps matching, where we optimize first rigidly and then using only diffeomorphisms with LDDMM, by better choosing the parameters. However, we want to highlight with this example that the choice of parameters for this 2-steps optimization, and in particular the choice of the kernel, depends on the first rigid alignment we obtain. This can be very bad, specially since the rigid alignment could be only partial, and is unknown at the beginning of the optimization. However, with our approach, the choice of kernel only depends on the source image, since in this model the kernel is transported by the rigid deformations and scalings (or equivalently because in problem (8.17) we apply the deformation induced by the diffeomorphisms first and then we apply rigid deformation and the scaling).

These examples highlight the importance of performing rigid and diffeomorphic registrations jointly. Moreover, by adjusting the parameters, one can favor solutions that require more or less deformation.

# Chapter 9

# Anisotropy II : The group of automorphisms and transport of metrics

## Contents



This chapter presents work that is currently in progress. As in chapter 8, the goal is to track and model the anisotropy of shapes. Here we directly consider the shape space as the space of Riemannian metrics of finite regularity on a finite dimensional compact manifold $M$. Given an ambient Riemannian metric $g^M$ of $M$, the anisotropy is therefore characterized by comparing any metric $g$ to this reference metric $g_M$.

The natural group acting on this space of metrics is the group of $C^k$ automorphisms of the tangent bundle $TM$, which can transport metrics via pullback. This gives a natural extension of classical density-based models in geometric analysis and computational anatomy. We prove in section 9.1 that the group of automorphisms carries a structure of Banach half-group. Moreover we also introduce the natural half-Lie subgroup of automorphisms preserving the ambient metric $g^M$, which will allow to follow the anisotropy of metrics with regards to $g^M$.

In section 9.2, we study the differentiability of the action on metrics. We will see in particular that this action does not satisfy the (S.1-3) conditions of 6.1, and the general results of chapter 6 do not directly apply. Nevertheless, we can recover existence and uniqueness of solutions of the metric transport equation induced by the infinitesimal action of the automorphisms group.

Finally, we introduce a variational problem induced by the pullback of the Ebin metric. We propose several numerical examples, based on a triangular discretization of metrics, which implements the action of automorphisms, and solve this variational problem. This gives rise to a new setting to explore the role of anisotropy, frame alignment, and deformation flows, that we hope to continue in future researches.

## 9.1 The half-Lie group of automorphisms of the tangent bundle

In this section we introduce the group of $C^k$ automorphisms of the tangent bundle of a finite dimensional compact manifold $M$ and we study its differential structure and properties. We start by actually considering morphisms of general vector bundles over $M$ and we prove it is a Banach manifold. The main ingredient is to re-write this space as a space of $C^k$ sections of a fibered bundle over $M$ and then to use [80]. The space of automorphisms is then simply an open subset of this Banach manifold.

We then suppose the manifold $M$ is equipped with an ambient Riemannian metric $g^M$, allowing to define the subgoup of automorphisms that preserve this metric. We prove that this new group is also a Banach half-Lie group and we investigate the general evolution equation (4.7) on this group.

### 9.1.1 Automorphisms of a general vector bundle

In this section, we define the group of automorphisms of a vector bundle, and we adapt the proof of [16, 1] to define a differential structure on this group and to show it is a half-Lie group.

#### 9.1.1.1 Vector bundle morphisms

Let $M$ be a finite dimensional compact manifold, of dimension $d$, and $p : E \to M$ a vector bundle over $M$ of dimension $n$. We denote by $\mathrm{Hom}_{C^k}(E)$ the space of $C^k$-morphisms of $E$, i.e. the space of $C^k$ fiber-preserving maps $A : E \to E$ such that the restriction of $A$ to each fiber is linear. Note that for $A \in \mathrm{Hom}_{C^k}(E)$, there exists a unique $C^k$-map $\underline{A} : M \to M$ such that $p \circ A = \underline{A} \circ p$.

$$\begin{tikzcd} E \arrow[r, "A"] \arrow[d, "p"] & E \arrow[d, "p"] \\ M \arrow[r, "\underline{A}"] & M \end{tikzcd}$$

In the next proposition, we define a differential structure on $\mathrm{Hom}_{C^k}(E)$.

**Proposition 9.1** (Differential structure of $\mathrm{Hom}_{C^k}(E)$).

1. *The space* $\mathrm{Hom}_{C^k}(E)$ *is a Banach manifold.*
2. *The projection*
$$\begin{array}{ccc} \mathrm{Hom}_{C^k}(E) & \longrightarrow & C^k(M,M) \\ A & \longmapsto & \underline{A} \end{array}$$
*is smooth.*
3. *Let* $l \geq 0$*, then the composition*
$$\begin{array}{ccc} \mathrm{Hom}_{C^{k+l}}(E) \times \mathrm{Hom}_{C^k}(E) & \longrightarrow & \mathrm{Hom}_{C^k}(E) \\ (A,B) & \longmapsto & A \circ B \end{array}$$
*is* $C^l$*, and* $C^\infty$ *for the first variable for* $B$ *fixed*
4. *The evaluation*
$$\begin{array}{ccc} \mathrm{Hom}_{C^k}(E) \times E & \longrightarrow & E \\ (A,v) & \longmapsto & A(v) \end{array}$$
*is* $C^k$

*Proof.* We denote by $L(E,E)$ the set of all linear mappings between the fibers i.e.

$$L(E,E) = \bigsqcup_{(x,y)\in E\times E} L(E_x, E_y).$$

We denote also by $s,t : L(E,E) \to M$ the source and target mappings. The triple $(L(E,E), s\times t, M\times M)$ is a vector bundle over $M\times M$ with fibers isomorphic to $L(\mathbb{R}^n,\mathbb{R}^n)$. Moreover we can see that the triple $(L(E,E), s, M)$, that we will denote by $L_1(E,E)$, also defines a fiber bundle (but not a vector bundle anymore) over $M$, such that the fiber over $x \in M$ is the union

$$L(E_x, E) := \bigsqcup_{y\in M} L(E_x, E_y)$$

1. *Differential structure of* $\mathrm{Hom}_{C^k}(E)$. We have the canonical identification [80, 1]:

$$\mathrm{Hom}_{C^k}(E) \simeq \Gamma_{C^k}(L_1(E,E)).$$

Indeed, for $A \in \mathrm{Hom}_{C^k}(E)$, and $x \in M$, we can consider the linear restriction $A_x : E_x \to E_{\underline{A}(x)}$ of $A$ along the fiber $E_x$. This gives rise to a corresponding section of $L_1(E,E)$ given by

$$\Gamma_A : M \to L(E,E), \Gamma_A(x) = A_x$$

and we have in particular that $\Gamma_A \in \Gamma_{C^k}(L_1(E,E))$. Conversely for $\Gamma \in \Gamma_{C^k}(L_1(E,E))$, we can define the corresponding $C^k$ vector bundle morphism $A \in \mathrm{Hom}_{C^k}(E)$ with

$$\underline{A}(x) = t \circ \Gamma(x),\ x \in M, \qquad Av = \Gamma\left(p(v)\right) v,\ v \in E.$$

Therefore the space $\mathrm{Hom}_{C^k}(E)$ is a Banach manifold ([80, §.12] and [34, §.6], note that in [34] the space of vector bundle morphisms $\mathrm{Hom}_{C^k}(E)$ can be viewed as a vector bundle over $C^k(M,M)$ by seeing that $\mathrm{Hom}_{C^k}(E) = \bigsqcup_{f\in C^k(M,M)} L(E, f^*E)$).

For $A \in \mathrm{Hom}_{C^k}(E)$, we will denote by $\Gamma_A$ the corresponding section in $\Gamma_{C^k}(L_1(E,E))$. Let's clarify a bit more the differential structure on $\Gamma_{C^k}(L_1(E,E))$ and explain how we can do differentiable calculus. Let $\Gamma_A \in \Gamma_{C^k}(L_1(E,E))$ corresponding to the morphism $A \in \mathrm{Hom}_{C^k}(E)$. Then [80, 12.6] there exists a vector bundle neigborhood of $\Gamma_A$ in $\Gamma_{C^k}(L_1(E,E))$, this means there exists a subbundle $\xi \to M$ over $M$ of $L_1(E,E)$, such that $\xi$ is open in $L_1(E,E)$ and $\Gamma_A \in \Gamma_{C^k}(\xi)$. Moreover, $\xi$ can be equipped with a vector bundle structure over $M$ such that $\Gamma_{C^k}(\xi)$ is a Banach vector space. This construction defines the differential structure of $\Gamma_{C^k}(L_1(E,E))$, and $\Gamma_{C^k}(\xi)$ is an open chart of $\Gamma_{C^k}(L_1(E,E))$. Moreover, the compact manifold $M$ can be covered by a finite number of compact submanifolds $M_1, ..., M_k$ such that each vector bundle $\xi \to M_i$ is a trivialization, meaning that there exists a smooth vector bundle isomorphism $\phi_i$, such that $\underline{\phi_i} : M_i \to \bar{\mathbb{B}}_1$ with $\bar{\mathbb{B}}_1$ the unit ball of $\mathbb{R}^d$ is an diffeomorphism and

$$\begin{array}{ccc}
\xi|_{M_i} & \xrightarrow{\phi_i} & \bar{\mathbb{B}}_1 \times (\mathbb{R}^n \times L(\mathbb{R}^n, \mathbb{R}^n)) \\
\Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \pi_1} \\
M_i & \xrightarrow{\underline{\phi_i}} & \bar{\mathbb{B}}_1
\end{array}$$

The Banach space $\Gamma_{C^k}(\xi)$ can be identified [80, §.4] to the space

$$\Gamma_{C^k}\left(\xi; (M_i, \phi_i)_i\right) = \{(f_1, ..., f_k) \in \prod_{i \leq k} C^k(\bar{\mathbb{B}}_1, \mathbb{R}^n \times L(\mathbb{R}^n, \mathbb{R}^n)) \mid \phi_i^{-1} f_i \underline{\phi_i}_{|M_i \cap M_j} = \phi_j^{-1} f_j \underline{\phi_j}_{|M_i \cap M_j}\},$$

which is a closed vector subspace of the Banach space $\prod_{i \leq k} C^k(\bar{\mathbb{B}}_1, \mathbb{R}^n \times L(\mathbb{R}^n, \mathbb{R}^n))$, through the smooth isomorphism

$$\psi : \left\{ \begin{array}{ccc} \Gamma_{C^k}(\xi) & \longrightarrow & \Gamma_{C^k}\left(\xi; (M_i, \phi_i)_i\right) \\ \Gamma & \longmapsto & (\phi_1 \Gamma \underline{\phi_1}^{-1}, ..., \phi_k \Gamma \underline{\phi_k}^{-1}) \end{array} \right.$$

This is particularly useful since calculus in the space $C^k(\bar{\mathbb{B}}_1, \mathbb{R}^n \times L(\mathbb{R}^n, \mathbb{R}^n))$ is well known, as we will see in the second part of this proof.

*2. Projection on $C^k(M,M)$.* Let's prove now the second point. Note that using the identification $\mathrm{Hom}_{C^k} \simeq \Gamma_{C^k}(L_1(E,E))$, we have that for $A \in \mathrm{Hom}_{C^k}(E)$,

$$\underline{A} = t \circ \Gamma.$$

Therefore it is actually equivalent to prove that the pushforward of the sections of $L_1(E,E)$ by $t$

$$t_* : \left\{ \begin{array}{ccc} \Gamma_{C^k}(L_1(E,E)) & \longrightarrow & C^k(M,M) \\ \Gamma_A & \longmapsto & t \circ \Gamma_A \end{array} \right.$$

is smooth. This is straightforward using [80, theorem 13.5], and because the target mapping $t : L_1(E,E) \to M$ is smooth (here we have the immediate identification $C^k(M,M) \simeq \Gamma_{C^k}(M \times M, \mathrm{pr}_1)$).

*3. Differentiability of the composition.* We continue now with the third point. Let $l \geq 0$. We define the manifold

$$L(E,E)^{[2]} = \{(A,B) \in L(E,E) \times L(E,E) \mid s(A) = t(B)\}.$$

Note this defines a fiber bundle over $M$

$$\begin{array}{ccc} L(E,E)^{[2]} & \longrightarrow & M \\ (A,B) & \longmapsto & s(B) \end{array}$$

with fiber given by $\bigsqcup_{y \in M} L(E_y, E) \times L(E_x, E_y)$ over $x \in M$. The space $L(E,E)^{[2]}$ describes the set of composable pairs of $L(E,E)$ and we can now introduce the fiberwise composition

$$\mathrm{Comp} : \left\{ \begin{array}{ccc} L(E,E)^{[2]} & \longrightarrow & L_1(E,E) \\ (A_{y,z}, B_{x,y}) & \longmapsto & A_{y,z} \circ B_{x,y} \end{array} \right.$$

We can see that the mapping Comp is a smooth fiber bundle morphism. By [80, theorem 13.5], the pushforward $\mathrm{Comp}_* : \Gamma_{C^k}(L(E,E)^{[2]}) \to \Gamma_{C^k}(L_1(E,E))$ is also smooth. Let $A, B \in \Gamma_{C^k}(L_1(E,E))$, the composition $A \circ B$ is given as

$$\Gamma_{A \circ B} = \mathrm{Comp}_*((\Gamma_A \circ t \circ \Gamma_B) \times \Gamma_B)$$

It remains to prove that the mapping

$$\begin{array}{ccc} \Gamma_{C^{k+l}}(L_1(E,E)) \times \Gamma_{C^k}(L_1(E,E)) & \longrightarrow & \Gamma_{C^k}(L(E,E)^{[2]}) \\ (\Gamma_A, \Gamma_B) & \longmapsto & (\Gamma_A \circ \ t \circ \Gamma_B) \times \Gamma_B \end{array}$$

is $C^l$. First, by Bastiani/Palais composition theorem on the compact manifold $M$ (see [80, 59]), we get that, for any finite dimensional manifold $F_0$, the map

$$C^{k+l}(M, F_0) \times C^k(M, M) \to C^k(M, F_0), \quad (f, \phi) \mapsto f \circ \phi$$

is $C^l$. Therefore using charts of $L_1(E,E)$, and the fact that the map $\Gamma \mapsto t \circ \Gamma$ is smooth (by the second point) from $\Gamma_{C^k}(L_1(E,E))$ to $C^k(M,M)$, we get

$$\begin{array}{ccc} \Gamma_{C^{k+l}}(L_1(E,E)) \times \Gamma_{C^k}(L_1(E,E)) & \longrightarrow & C^k(M, L(E,E)) \\ \Gamma_A, \Gamma_B & \longmapsto & \Gamma_A \circ (t \circ \Gamma_B) \end{array}$$

Note that this map is exactly the cause for the lack of regularity, since all the other maps are smooth and some morphisms between fiber bundles over $M$. Now we still need to come back to the bundle $L(E,E)^2$. We define another vector bundle $F \to M$ with fiber over $x \in M$ given by

$$F_x \coloneqq \bigsqcup_{y,y',z} L(E_{y'}, E_z) \oplus L(E_x, E_y) = L(E,E) \times L_1(E,E)_x$$

In other words, the bundle $F$ is the fiber product over $M$ of the bundle $s : L_1(E,E) \to M$ and of the trivial bundle $M \times L(E,E) \to M$, i.e. $F = (M \times L(E,E)) \times_M L_1(E,E)$. By [80, theorem 13.7], we therefore have

$$\Gamma_{C^k}(F) = C^k(M, L(E,E)) \times \Gamma_{C^k}(L_1(E,E))$$

and now the map

$$\Xi : \left\{ \begin{array}{ccc} \Gamma_{C^{k+l}}(L_1(E,E)) \times \Gamma_{C^k}(L_1(E,E)) & \longrightarrow & \Gamma_{C^k}(F) \\ \Gamma_A, \Gamma_B & \longmapsto & (\Gamma_A \circ t \circ \Gamma_B, \Gamma_B) \end{array} \right.$$

is $C^l$. We are going to co-restrict this map using an open subbundle of $F$, which will allow us to go back to $\Gamma_{C^k}(L(E,E)^{[2]})$. We can equip $M$ with a smooth Riemannian metric, and $E$ with a connection (or equivalently a parallel transport). Since $M$ is compact, it has a positive (global) convexity radius $r_{\text{conv}}(M) > 0$. Fix $0 < \varepsilon < r_{\text{conv}}(M)$. In particular, for any $y, y' \in M$ such that $d(y, y') < \varepsilon$ (with $d$ the Riemannian distance), there exists a unique minimizing geodesic between $y$ and $y'$ given by the exponential map. We denote by $P_{y' \leftarrow y} : E_y \to E_{y'}$ the parallel transport over this geodesic, which therefore depends smoothly on $(y, y')$. We now define the open subbundle $F_\varepsilon \subset F$

$$F_\varepsilon := \{(A, B) \in F \mid d\,(s(A), t(B)) < \varepsilon\}.$$

Now we clearly see that the image of the map $\Xi$ is included $\Gamma_{C^k}(F_\varepsilon)$, since for any $(\Gamma_A, \Gamma_B) \in \Gamma_{C^{k+l}}(L_1(E, E)) \times \Gamma_{C^k}(L_1(E, E))$, we have

$$s \circ \Gamma_A \circ t \circ \Gamma_B = t \circ \Gamma_B$$

Thus the map $\Xi$ co-restricts to a $C^k$ map from $\Gamma_{C^{k+l}}(L_1(E, E)) \times \Gamma_{C^k}(L_1(E, E))$ to $\Gamma_{C^k}(F_\varepsilon)$. Moreover, the parallel transport of $E$ induces a smooth bundle morphism

$$\Theta : \left\{ \begin{array}{ccc} F_\varepsilon & \longrightarrow & L(E, E)^{[2]} \\ (A, B) & \longmapsto & (A \circ P_{s(A) \leftarrow t(B)}, B) \end{array} \right. ,$$

which in turn induces by [80, theorem 13.5] a smooth morphism

$$\Theta_* : \Gamma_{C^k}(F_\varepsilon) \to \Gamma_{C^k}(L(E, E)^{[2]})\,.$$

We summarize all these maps in the following diagram

$$\Gamma_{C^{k+l}}(L_1(E, E)) \times \Gamma_{C^k}(L_1(E, E)) \xrightarrow{\Xi} \Gamma_{C^k}(F_\varepsilon) \xrightarrow{\Theta_*} \Gamma_{C^k}(L(E, E)^{[2]}) \xrightarrow{\text{Comp}_*} \Gamma_{C^k}(L_1(E, E))$$

We can now conclude, by composing all this maps, that the composition law $A, B \mapsto A \circ B$ is indeed $C^l$:

$$\Gamma_{A \circ B} = \text{Comp}_* \circ \Theta_* \circ \Xi(\Gamma_A, \Gamma_B).$$

Similarly, if we fix $B \in \text{Hom}_{C^k}(E)$, the composition on the left

$$\begin{array}{ccc} \text{Hom}_{C^k}(E) & \longrightarrow & \text{Hom}_{C^k}(E) \\ A & \longmapsto & A \circ B \end{array}$$

is the map $\Gamma_A \mapsto \text{Comp}_*((\Gamma_A \circ t \circ \Gamma_B) \times \Gamma_B)$ which is smooth, because the map $\Gamma_A \mapsto (\Gamma_A \circ t \circ \Gamma_b) \times \Gamma_B$ is now smooth too.

*4. Differentiability of the evaluation.* We now turn to the proof of the last point. Note that we have a smooth inclusion $\Gamma_{C^k}(L_1(E, E)) \subset C^k(M, L(E, E))$ given by the lift of the fiberwise bundle morphism

$$\begin{array}{ccc} L_1(E, E) & \longrightarrow & M \times L(E, E) \\ A & \longmapsto & (s(A), A) \end{array} .$$

Moreover, for any $x \in M$, the evaluation on $C^k(M, L(E, E))$ at $x$

$$C^k(M, L(E, E)) \to L(E, E),$$

is smooth (cf. [80, 59]). In particular, the map

$$\begin{array}{ccc} \text{Hom}_{C^k}(E) \times E & \longrightarrow & L_1(E, E) \times_M E \\ (A, v) & \longmapsto & (A(p(v)), v) \end{array}$$

is $C^k$, as the co-resctriction of a $C^k$ map to a finite dimension submanifold. Since the fiberwise evaluation in $L_1(E, E) \times_M E$ is smooth, the last point follows from [80, theorem 13.5]. $\square$

#### 9.1.1.2 The group of automorphisms of $E$

We now define the space of automorphisms $\mathrm{Aut}_{C^k}(E) \subset \mathrm{Hom}_{C^k}(E)$ as the space of invertible morphisms $A : E \to E$, which means that there must exist a unique morphism $A' \in \mathrm{Hom}_{C^k}(E)$ such that $A \circ A' = A' \circ A = \mathrm{Id}_E$. This directly implies that $\underline{A} : M \to M$ is a $C^k$ diffeomorphism of $M$. We get the following result :

**Proposition 9.2** (The group of automorphisms)**.** *The space* $\mathrm{Aut}_{C^k}(E)$ *of* $C^k$ *automorphisms of* $E$ *is a Banach half-Lie group. Suppose moreover* $E$ *is equipped with a connection. Its Lie algebra is given by the isomorphism*

$$\mathfrak{aut}_{C^k}(E) \simeq \Gamma_{C^k}(TM \times_M L(E,E)) \simeq \Gamma_{C^k}(TM) \oplus \mathrm{End}_{C^k}(E)$$

*where* $TM \times_M L(E,E)$ *is the vector bundle over* $M$ *with fiber at* $x \in M$ *given as* $T_xM \oplus L(E_x, E_x)$*, and* $\mathrm{End}_{C^k}(E)$ *is the space of* $C^k$*-vector morphisms of* $E$ *that covers the identity. The derivative of the right multiplication at identity is then given by*

$$\begin{array}{ccc} \mathfrak{aut}_{C^k}(E) \times \mathrm{Aut}_{C^k}(E) & \longrightarrow & T\,\mathrm{Aut}_{C^k}(E) \\ (u,a), B & \longmapsto & (u \circ \underline{B}, a \circ B) \end{array}$$

*Moreover, its space of* $C^l$ *elements is exactly* $\mathrm{Aut}_{C^{k+l}}(E)$ *and the family* $\{\mathrm{Aut}_{C^k}(E),\ k \in \mathbb{N}_{>0}\}$ *satisfies the (G.1-5) conditions of section 4.3.*

*Proof.* Recall that the mapping

$$t_* : \left\{ \begin{array}{ccc} \Gamma_{C^k}(L_1(E,E)) & \longrightarrow & C^k(M,M) \\ \Gamma_A & \longmapsto & t \circ \Gamma_A \end{array} \right.$$

is smooth. We denote by $\widetilde{\mathrm{Diff}}_{C^k}(M)$ the image of automorphisms space $\mathrm{Aut}_{C^k}(E)$ by the mapping $t_*$. This corresponds to the space of diffeomorphisms of $M$ that can be lifted to a $C^k$ automorphisms of $E$. Note that if two $C^k$ diffeomorphisms $\varphi$ and $\psi$ are $C^k$ homotopic, then $\varphi^{-1} \circ \psi$ is $C^k$ homotopic to $Id$. Now this $C^k$ homotopy allows to lift $\varphi^{-1} \circ \psi$ to an automorphism of $E$ using parallel transport. This means that $\varphi$ can be lifted to an automorphism of $E$ if and only if $\psi$ can be lifted. Because $C^k(M,M)$ is modeled on a locally convex space and since $\mathrm{Diff}_{C^k}(M)$ is an open subset of $C^k(M,M)$ [16], every diffeomorphism $\varphi$ of $M$ belongs to an open subset of diffeomorphisms that are $C^k$ homotopic to $\varphi$. Therefore the space $\widetilde{\mathrm{Diff}}_{C^k}(M)$ is open in $C^k(M,M)$. Moreover, note that the mapping

$$F : \left\{ \begin{array}{ccc} \mathrm{Hom}_{C^k}(E) & \longrightarrow & \mathbb{R} \\ \Gamma_A & \longmapsto & \inf_{x \in M} \det \Gamma_A(x) \end{array} \right.$$

is continuous. Indeed it is the composition of the mapping $\det_* : \Gamma_A \mapsto \det \circ \Gamma_A$ which is smooth (because the determinant is smooth on the fibers) and the mapping defined on $C^k(M,\mathbb{R})$ by $f \mapsto \inf_{x \in M} f(x)$. Now since $M$ is compact, the space of $C^k$ automorphisms is exactly

$$\mathrm{Aut}_{C^k}(E) = t_*^{-1}(\widetilde{\mathrm{Diff}}_{C^k}(M)) \cap F^{-1}(\mathbb{R}_{>0}).$$

Since the space of automorphisms is open in $\mathrm{Hom}_{C^k}(E)$, this proves that $\mathrm{Hom}_{C^k}(E)$ is therefore a Banach manifold. By [80, theorem 13.6], the tangent space at $\Gamma_A$ is given by $T_{\Gamma_A} \mathrm{Hom}_{C^k}(E) = \Gamma_{C^k}((\Gamma_A)^* V L_1(E,E))$ where $VL_1(E,E)$ denotes the vertical bundle over $L_1(E,E)$ (whose fiber at $\alpha_x \in L_1(E_x,E)$ is $V_{\alpha_x}L_1(E,E) = T_{\alpha_x}(L_1(E_x,E))$). In particular, the Lie algebra of $\mathrm{Hom}_{C^k}(E)$ is then $T_{\Gamma_{\mathrm{Id}_E}} \mathrm{Hom}_{C^k}(E) \simeq \Gamma_{C^k}((\Gamma_{\mathrm{Id}_E})^* VL_1(E,E))$. Let's explicit more the Lie algebra, in particular since we supposed $E$ is equipped with a smooth connection. This induces a smooth connection also on the bundle $(L(E,E), s \times t, M \times M)$. Equivalently, this implies a Ehresmann connection, i.e. we have the split decomposition

$$TL(E,E) \simeq VL(E,E) \oplus HL(E,E)$$

But then we also have the immediate vector bundle isomorphism

$$\begin{array}{ccc} TL(E,E) & \xrightarrow{\sim} & (s\times t)^*\left((TM \times TM) \times L(E,E)\right) \\ \alpha & \mapsto & (d(s\times t)(\mathrm{pr}_H(\alpha)), \mathrm{pr}_V(\alpha)) \end{array} \tag{9.1}$$

where $\mathrm{pr}_H$ (resp. $\mathrm{pr}_V$) is the smooth projection on $HL(E,E)$ (resp. $VL(E,E)$). Moreover the bundle $VL_1(E,E)$ over $L(E,E)$ is the vector subbundle of $TL(E,E)$ defined by

$$VL_1(E,E) = \ker(ds)$$

Through the isomorphism (9.1), we then see that, for $\alpha_x \in L(E,E)$, we have $V_{\alpha_x}L_1(E,E) \simeq T_yM \oplus L(E_x,E_y)$ where $y = t(\alpha_x)$, and this defines an isomorphism of vector bundles. Therefore we finally get the identification $(\Gamma_{\mathrm{id}_E})^* VL_1(E,E) \simeq TM \times_M L(E,E)$ so that, using [80, theorem 13.5]

$$\mathfrak{aut}_{C^k}(E) \simeq \Gamma_{C^k}(TM) \oplus \mathrm{End}_{C^k}(E).$$

Let us now prove the last point, starting by proving that

$$(\mathrm{Aut}_{C^k}(E))^l = \mathrm{Aut}_{C^{k+l}}(E).$$

We adapt the proof of [16]. We already saw that $\mathrm{Aut}_{C^{k+l}}(E) \subset (\mathrm{Aut}_{C^k}(E))^l$ since the composition mapping

$$\begin{array}{ccc} \mathrm{Aut}_{C^{k+l}}(E) \times \mathrm{Aut}_{C^k}(E) & \longrightarrow & \mathrm{Aut}_{C^k}(E) \\ A, B & \longmapsto & A \circ B \end{array}$$

is $C^l$. Conversely, let $A \in (\mathrm{Aut}_{C^k}(E))^l$ and let's show that $A \in \mathrm{Aut}_{C^{k+l}}(E)$. Let $x \in M$, $y \in T_xM$ and $U \subset M$ a local trivialization of $L_1(E,E)$ around $x$. We can using again [80, §.4] reduce to restrictions to $U$ and work in $\mathrm{Aut}_{C^k}(U \times \mathbb{R}^n)$. Let $h \in T_{\mathrm{id}} \mathrm{Aut}_{C^k}(U \times \mathbb{R}^n) \simeq \Gamma_{C^k}(U \times \mathbb{R}^n \times L(\mathbb{R}^n, \mathbb{R}^n))$ such that $h$ is constant and for $x' \in U$, $h(x') = (x', y, \mathrm{id})$ (this means $h$ is constant and $\underline{h}(x') = y$). Denote by $h_t$ the mapping such that $h_t(x') = (x', ty, \mathrm{id})$. Now, for $t \in \mathbb{R}$ sufficiently small, the mapping $id + h_t$ is in $\mathrm{Aut}_{C^k}(U \times \mathbb{R}^n)$ (because the set of automorphisms is open as we saw previously). The mapping

$$\begin{array}{ccc} ]-\varepsilon, \varepsilon[ & \longrightarrow & U \times \mathbb{R}^n \times L(\mathbb{R}^n, \mathbb{R}^n) \\ t & \longmapsto & \frac{d^k}{ds^k}\left([\Gamma_A \circ (\mathrm{id} + \underline{h_t})](x+sy)\right)_{s=0} \end{array}$$

is $C^l$ as a composition of $t \mapsto \Gamma_A \circ (\mathrm{id} + \underline{h_t}) = A \circ (\mathrm{id} + h)$ which is $C^l$ by definition of $A$, and $B \in \Gamma_{C^k}(U \times \mathbb{R}^n \times L(\mathbb{R}^n, \mathbb{R}^n)) \mapsto d^k B(x)(y,..,y)$ which is smooth. But then we see that $\frac{d^k}{ds^k}\left([\Gamma_A \circ (\mathrm{id} + \underline{h_t})](x+sy)\right)_{s=0} = \frac{d^k}{ds^k}\left(\Gamma_A(x+sy+ty)\right)_{s=0} = d^k\Gamma_A(x+ty)(y,...y)$, and therefore, $\Gamma_A$ is also a $C^l$ section of $U \times \mathbb{R}^n \times L(\mathbb{R}^n, \mathbb{R}^n)$. This finishes to prove that $(\mathrm{Aut}_{C^k}(E))^l = \mathrm{Aut}_{C^{k+l}}(E)$.

It remains to prove that the family $\{\mathrm{Aut}_{C^k}(E),\ k \in \mathbb{N}_{>0}\}$ is an admissible graded group structure as defined in [82]. The fact that $\mathrm{Aut}_{C^{k+l}}(E) \subset \mathrm{Aut}_{C^k}(E)$ with smooth inclusion is a consequence of [80, theorem 5.1]. Let $u \in \mathfrak{aut}_{C^{k+l}}(E)$, and $B \in \mathrm{Aut}_{C^k}(E)$. In charts adapted to the bundle $L_1(E, E)$, the derivative of the composition $\partial_A(A \circ B)_{A=\mathrm{id}}(u)$ is given simply by $\mathrm{Comp}_*((\underline{u} \circ \underline{B} \times B)$. Similarly to the proof of the the third point of proposition 9.1, this defines a $C^l$ mapping, which is smooth for the first variable when $B$ is fixed. The fact that $(A, u) \mapsto A \circ u := \partial_B(A \circ B)_{B=u}$, for $A \in \mathrm{Aut}_{C^{k+1}}(E)$ and $u \in \mathfrak{aut}_{C^k}(E)$, is $C^1$ is also straightforward in charts. □

**Remark 9.3** (Lie algebra of the automorphisms using the Atiyah algebroid)**.** *Note here that the isomorphism between the Lie algebra and $\Gamma_{C^k}(TM) \oplus \mathrm{End}_{C^k}(E)$ is not canonical and is dependent of the connection we choose on $E$. Similarly, the computation of the derivative of the right multiplication at identity is also dependent of this choice of connection.*

*Following [62, 57], this decomposition of the space $\mathfrak{aut}_{C^k}(E)$ can be seen using the Atiyah splitting sequence. Denote by $\mathfrak{gl}(E)$ the vector bundle over $M$ of endomorphisms of $E$, and $\mathrm{At}(E)$ the Atiyah Lie algebroid of derivations of $E$ (cf. for example [12, 62]), that is to say the vector bundle over $M$ whose space of sections consists of first-order derivations of $E$, i.e. linear operators $D : \Gamma(E) \to \Gamma(E)$ satisfying the Leibniz rule*

$$D(fs) = fD(s) + X \cdot fs, \quad \forall f \in C^\infty(M),\ s \in \Gamma(E)$$

*for a uniquely determined $X \in \Gamma(TM)$. In particular, we get the identification [57]*

$$\mathfrak{aut}_{C^k}(E) = \Gamma_{C^k}(\mathrm{At}(E))$$

*Moreover, we get the following exact sequence*

$$0 \to \mathfrak{gl}(E) \to \mathrm{At}(E) \to TM \to 0$$

*The splitting of this sequence is then equivalent to the definiton of a connection $\nabla : TM \to \mathrm{At}(E)$ on $E$, and thus the following decomposition yields*

$$\mathfrak{aut}_{C^k}(E) = \Gamma_{C^k}(\mathrm{At}(E)) \simeq \Gamma_{C^k}(TM) \oplus \mathrm{End}(E).$$

**Remark 9.4.** *This directly implies that the inverse mapping*

$$\mathrm{inv} : \begin{cases} \mathrm{Aut}_{C^{k+l}} & \longrightarrow & \mathrm{Aut}_{C^k} \\ A & \longmapsto & A^{-1} \end{cases}$$

*is $C^l$ by using the implicit function theorem in Banach spaces [61].*

#### 9.1.1.3 Flows of automorphisms

We finish this part with regularity properties of the group $\mathrm{Aut}_{C^k}(E)$, and how we can integrate curves in the Lie algebra. We suppose the vector bundle $E \to M$ is equipped with a smooth connection, that we denote $\nabla^E$. By [82], if $\alpha_t \in L^p(I, \mathfrak{aut}_{C^{k+1}}(E))$ is a curve in the Lie algebra of $\mathrm{Aut}_{C^{k+1}}(E)$, we can integrate it and define an absolutely continuous curve $\Psi_t \in AC_{L^p}(I, \mathrm{Aut}_{C^k}(E))$ solution of

$$\partial_t \Psi_t = T_{\mathrm{id}_E} R_{\Psi_t}(\alpha_t), \quad \Psi_0 = \mathrm{id}_E \tag{9.2}$$

As stated before, we have the isomorphism

$$\mathfrak{aut}_{C^k}(E) \simeq \Gamma_{C^k}(TM) \oplus \mathrm{End}_{C^k}(E),$$

which is not canonical and dependent of the connection $\nabla^E$ on the bundle $E \to M$. In particular, using this identification, this leads to a curve $\alpha_t = (u_t, a_t) \in L^p(I, \Gamma_{C^k}(TM) \oplus \mathrm{End}_{C^k}(E))$. We will use this identification to interpret geometrically equation (9.2). We start by computing the derivative of the evaluation

$$\mathrm{Ev}_v : A \mapsto A(v)$$

where $v \in E$.

**Lemma 9.5** (Differential of the evaluation)**.** *Let $v \in E$, and $A \in \mathrm{Aut}_{C^k}(E)$. The differential of the evaluation at $A$ is given by*

$$T_A \,\mathrm{Ev}_v(u \circ \underline{A}, a \circ A) = (u(\underline{A}(x)), aA(v)) \tag{9.3}$$

*where $x = p(v) \in M$, and we use the identification $T_{Av}E \simeq T_{\underline{A}(x)}M \times E_{Av}$ given by the connection.*

*Proof.* This follows from the fact that, for $v \in E$ fixed, with $x = p(v)$, the evaluation $\mathrm{Ev}_v$ is the composition of the smooth map

$$\begin{array}{ccc} \mathrm{Hom}_{C^k}(E) & \longrightarrow & L(E_x, E) \\ A & \longmapsto & A_x \end{array},$$

as seen in the proof of proposition 9.1.1, and of the fiberwise evaluation

$$\begin{array}{ccc} L(E_x, E) & \longrightarrow & E \\ A_x & \longmapsto & A_x v \end{array}.$$

Using [80, theorem 13.6], the connection on $E$ and the previous identification $T_{A_x}L(E_x, E) \simeq T_{\underline{A}(x)}M \oplus L(E_x, E_{\underline{A}(x)})$, the differential of these maps are

$$\left\{ \begin{array}{ccc} T_A \,\mathrm{Hom}_{C^k}(E) & \to & T_{A_x}L(E_x, E) \\ (u \circ \underline{A}, a \circ A) & \mapsto & (u(\underline{A}(x)), a \circ A) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} T_{A_x}L(E_x, E) & \to & T_{\underline{A}(x)}E \\ (u(\underline{A}(x)), a \circ A) & \mapsto & (u(\underline{A}(x)), aA(v)) \end{array} \right.$$

and this completes the proof. □

We can now formulate the main result:

**Proposition 9.6** (Flow in $\mathrm{Aut}_{C^k}(E)$)**.** *Let $\alpha_t = (u_t, a_t) \in L^p(I, \mathfrak{aut}_{C^{k+1}}(E))$, and $\Psi_t \in AC_{L^p}(I, \mathrm{Aut}_{C^k}(E)$ solution of (9.2), i.e.*

$$\partial_t \Psi_t = T_{\mathrm{id}_E} R_{\Psi_t}(\alpha_t), \quad \Psi_0 = \mathrm{id}_E \tag{9.4}$$

*Then for $v \in E_x$, the curve $V_t = \Psi_t v \in AC_{L^p}(I, E)$ along $x_t \in M$ satisfies :*

$$\partial_t x_t = u_t(x_t), \quad \nabla^E_{\frac{d}{dt}} V_t = a_t(V_t) \tag{9.5}$$

*Proof.* Let $x \in M$, $v \in E_x$ and we define the curves $x_t = \underline{\Psi}_t(x) \in M$ and $V_t = \Psi_t v = \mathrm{Ev}(\Psi_t, v) \in E$ so that

$$x_t = p(V_t)$$

Since $\Psi$ is absolutely continuous and the evaluation of automorphisms is smooth, the curves $x_t$ and $V_t$ are absolutely continuous. Using the splitting decomposition of $TE$, the derivative $\partial_t V_t \in T_{V_t}E$ is then simply

$$\partial_t V_t = (\partial_t x_t, \nabla^E_{\frac{d}{dt}} V_t)$$

Moreover, by lemma 9.5 and using (9.2), we have

$$\begin{aligned}
\partial_t V(t) &= T_{\Psi_t} \mathrm{Ev}_v(\dot{\Psi}_t) \\
&= T_{\Psi_t} \mathrm{Ev}_v \left(u \circ \underline{\Psi}_t, a_t \Psi_t\right) \\
&= (u \circ \underline{\Psi}_t(x), a_t \Psi_t(v)) \\
&= (u(x_t), a_t V_t)
\end{aligned}$$

Therefore the result follows from these identifications. $\square$

We will simply say that $\Psi_t$ is solution of

$$\frac{D}{dt}\Psi_t = a_t \circ \Psi_t, \quad \Psi_0 = \mathrm{id}_E \tag{9.6}$$

along the curve

$$\partial_t \underline{\Psi}_t = u_t \circ \underline{\Psi}_t, \quad \underline{\Psi}_0 = \mathrm{id}_M \tag{9.7}$$

### 9.1.2 Automorphisms of the tangent bundle

We suppose now $M$ is equipped with a $C^k$-Riemannian metric $g^M$. Note that the tangent bundle $TM$ of the manifold $M$ is itself a vector bundle over $M$. We denote by $\nabla$ the Levi-Civita connection associated with $g^M$. For any smooth vector field $u \in \Gamma_{C^{k+1}}(M)$ and any $h \in TM$, we will denote

$$\nabla(u)h \coloneqq \nabla_h u$$

so that

$$\nabla(u) : TM \to TM$$

is a $C^k$-morphism on $TM$ that is covering $\mathrm{id}_M$. In the rest of this part, we will therefore focus on the space of automorphisms of the tangent bundle $\mathrm{Aut}_{C^k}(TM)$. We saw in the previous section that this connection allows the identification

$$\mathfrak{aut}_{C^k}(TM) \simeq \Gamma_{C^k}(TM) \oplus \mathrm{End}_{C^k}(TM).$$

We also naturally have an inclusion of the group of $C^{k+1}$ diffeomorphisms of $M$ into the group of $C^k$ automorphisms using the differential :

**Proposition 9.7.** *The group of diffeomorphisms* $\mathrm{Diff}_{C^{k+1}}(M)$ *is included in* $\mathrm{Aut}_{C^k}(TM)$ *through the smooth mapping :*

$$d : \left\{ \begin{array}{ccc} \mathrm{Diff}_{C^{k+1}}(M) & \longrightarrow & \mathrm{Aut}_{C^k}(TM) \\ f & \longmapsto & df \end{array} \right.$$

*The differential mapping is moreover a group morphism, so that the group* $\mathrm{Diff}_{C^{k+1}}(M)$ *is a subgroup of* $\mathrm{Aut}_{C^k}(TM)$*. Moreover, the Lie algebra* $\Gamma_{C^{k+1}}(TM)$ *of* $\mathrm{Diff}_{C^{k+1}}(M)$ *is also included in the space* $\mathfrak{aut}_{C^k}(TM)$ *and we get*

$$d : \left\{ \begin{array}{ccc} \Gamma_{C^{k+1}}(TM) & \longrightarrow & \mathfrak{aut}_{C^k}(TM) \simeq \Gamma_{C^k}(TM) \oplus \mathrm{End}_{C^k}(TM) \\ u & \longmapsto & (u, \nabla u) \end{array} \right.$$

#### 9.1.2.1 Automorphisms of $TM$ that preserve a Riemannian metric

In this part, we also define the space of automorphisms that preserve $g^M$

$$\mathrm{Aut}_{C^k, g^M}(TM) := \{A \in \mathrm{Aut}_{C^k}(TM) \mid A_* g^M = g^M\}$$

where $A_* g^M$ is the pushforward of the metric $g^M$ by $A$ defined, for any $y \in M,\ h, k \in T_y M$, by

$$(A_* g)_y(h, k) = g_{\underline{A}^{-1}(y)}(A^{-1}h, A^{-1}k)$$

This space is a subgroup of the space of automophisms $\mathrm{Aut}_{C^k}(TM)$, and even a half-Lie subgroup :

**Proposition 9.8** (The subgroup $\mathrm{Aut}_{C^k, g^M}(TM)$)**.** *The space of automorphisms that preserve* $g^M$ *is a closed submanifold in the sense of [61], and a half-Lie subgroup of* $\mathrm{Aut}_{C^k}(TM)$*. Its Lie algebra is given by*

$$\mathfrak{aut}_{C^k, g^M}(TM) \simeq \Gamma_{C^k}(TM) \oplus \mathfrak{so}_{C^k, g^M}(TM)$$

*where* $\mathfrak{so}_{C^k, g^M}(TM) \subset \mathrm{End}_{C^k}(TM)$ *is the space of* $C^k$*-skew-symmetric endomorphisms of* $TM$ *with regards to* $g^M$*, i.e. the set of endomorphisms* $A \in \mathrm{End}_{C^k}(TM)$ *such that*

$$\forall h, k \in TM,\ g^M(Ah, k) + g^M(h, Ak) = 0. \tag{9.8}$$

*Moreover, the space* $\mathrm{Isom}_{C^{k+1}}(M, g^M)$ *of* $C^{k+1}$ *isometries of* $(M, g^M)$ *is embedded in* $\mathrm{Aut}_{C^k, g^M}(TM)$ *through the differential mapping*

$$d : \left\{ \begin{array}{ccc} \mathrm{Isom}_{C^{k+1}}(M, g^M) & \longrightarrow & \mathrm{Aut}_{C^k, g^M}(TM) \\ f & \longmapsto & df \end{array} \right.$$

**Remark 9.9.** *In this proposition, we use the term submanifold (also called sometimes split submanifold) in the sense of Lang [61], i.e. we assume that the tangent space of a submanifold is closed and admits a closed complementary space.*

*Proof.* The fact that the space of automorphisms that preserve the metric $g^M$ is a subgroup of $\mathrm{Aut}_{C^k}(TM)$ is straightforward. Moreover, we will see in the next sections that the action of automorphisms on the space of Riemannian metrics is continuous, so that the space $\mathrm{Aut}_{C^k, g^M}(TM)$ is also immediately a closed topological subgroup of

$\mathrm{Aut}_{C^k}(TM)$. We propose an other proof of this point along with the differentiable structure of $\mathrm{Aut}_{C^k,g^M}(TM)$ in the following.

We first define the space $\mathrm{Hom}_{C^k,g^M}(TM) \subset \mathrm{Hom}_{C^k}(TM)$ of vector bundle morphisms of $TM$ that preserves the metric $g^M$

$$\mathrm{Hom}_{C^k,g^M}(TM) \coloneqq \{A \in \mathrm{Hom}_{C^k}(TM) \mid \forall x \in M, A_x : T_xM \to T_{\underline{A}(x)}M \text{ is an isometry}\}.$$

We will prove that the space $\mathrm{Hom}_{C^k,g^M}(TM)$ identifies with the space of $C^k$-sections of a fiber bundle over $M$ and therefore acquires a Banach differentiable structure. We consider the subspace $\mathrm{Isom}(TM) \subset L(TM, TM)$ of isometries between tangent spaces

$$\mathrm{Isom}_{g^M}(TM, TM) \coloneqq \{(x, y, A) \in M \times M \times L(T_xM, T_yM) \mid \forall h, k \in T_xM, g_y(Ah, Ak) = g_x(h, k)\}$$

This space is naturally a subbundle of $L(TM, TM)$ over $M \times M$, and even aquires a structure of principal $O(n)$-bundle over $M \times M$ with fibers isomorphic

$$\mathrm{Isom}(T_xM, T_yM) = \{A \in L(T_xM, T_yM) \mid A \text{ is a linear isometry}\} \simeq O(n)$$

Moreover, considering only the source projection, the triple $(\mathrm{Isom}(TM, TM), s, M)$ also defines a fiber bundle over M, with fiber over $x \in M$ being the union

$$\mathrm{Isom}(T_xM, TM) = \bigsqcup_{y \in M} \mathrm{Isom}(T_xM, T_yM) \simeq M \times O(n).$$

We will denote this fiber bundle $\mathrm{Isom}_1(TM, TM)$ and we clearly see it is a closed subbundle of $L_1(TM, TM)$. Once again, we have a canonical identification between the sections of $\mathrm{Isom}_1(TM, TM)$ and the space $\mathrm{Hom}_{C^k,g^M}(TM)$ of vector bundle morphisms of $TM$ that preserves the metric $g^M$

$$\mathrm{Hom}_{C^k,g^M}(TM) \simeq \Gamma_{C^k}(\mathrm{Isom}_1(TM, TM))$$

This gives $\mathrm{Hom}_{C^k,g^M}(TM)$ a smooth Banach differentiable structure, and moreover it is a closed submanifold of $\mathrm{Hom}_{C^k}(TM)$ [80, 14.11]. Since $\mathrm{Aut}_{C^k,g^M}(TM) = \mathrm{Hom}_{C^k,g^M}(TM) \cap t_*^{-1}(\mathrm{Diff}_{C^k}(M))$, the space $\mathrm{Aut}_{C^k,g^M}(TM)$ is then an open subset of $\mathrm{Hom}_{C^k,g^M}(TM)$ and thus a Banach manifold, and a closed submanifold of $\mathrm{Aut}_{C^k}(TM)$. Therefore, the space $\mathrm{Aut}_{C^k,g^M}(TM)$ is a closed submanifold and a half-Lie subgroup of $\mathrm{Aut}_{C^k}(TM)$.

Moreover, its Lie algebra is given by

$$\mathfrak{aut}_{C^k,g^M}(TM) = \Gamma_{C^k}((\Gamma_{\mathrm{id}_{TM}})^* V \,\mathrm{Isom}_1(TM, TM))$$

Let us describe the bundle $(\Gamma_{\mathrm{id}_{TM}})^* V \,\mathrm{Isom}_1(TM, TM)$ over $M$. By definition, its fiber over $x \in M$ is given by

$$((\Gamma_{\mathrm{id}_{TM}})^* V \,\mathrm{Isom}_1(TM, TM))_x = T_{\mathrm{id}_{T_xM}} \mathrm{Isom}(T_xM, TM) \simeq T_xM \oplus T_{\mathrm{id}_{T_xM}} \mathrm{Isom}(T_xM, T_xM)$$

Since $T_{\mathrm{id}_{T_xM}} \mathrm{Isom}(T_xM, T_xM)$ is naturally the space $\mathfrak{so}(T_xM)$ of skew-adjoints endomorphisms of $T_xM$, we finally obtain the Lie algebra

$$\mathfrak{aut}_{C^k,g^M}(TM) = \Gamma_{C^k}(TM) \oplus \mathfrak{so}_{g^M}(TM).$$

□

**Remark 9.10.** *Note that this subgroup* $\mathrm{Aut}_{C^k,g^M}(TM)$ *is also isomorphic to the space of* $C^k$ *automorphisms* $\mathrm{Aut}_{C^k}(O(M))$ *of the orthonormal frame bundle* $O(M)$*, where*

$$O(M) = \bigsqcup_{x\in M} \{(e_1,\ldots,e_n) \mid (e_1,\ldots,e_n) \text{ is an orthonormal frame of } T_xM\},$$

*which is principal* $O(n)$*-bundle over* $M$*. Here, a* $C^k$ *automorphism* $\Psi \in \mathrm{Aut}_{C^k}(O(M))$ *of* $O(n)$ *is a* $O(M)-$*equivariant morphism* $\Psi : O(M) \to O(M)$*, i.e. such that*

$$\Psi(u\cdot A) = \Psi(u)\cdot A, \quad \forall u \in O(M),\, A \in O(n).$$

*Following [16, section 6], the group* $\mathrm{Aut}_{C^k}(O(M))$ *acquires a Banach half-Lie group structures, that is also isomorphic to* $\mathrm{Aut}_{C^k,g^M}(TM)$*.*

#### 9.1.2.2 Integration of flows in $\mathrm{Aut}_{C^k,g^M}(TM)$

We finish this section by integrating a special class of curves of $\mathfrak{aut}_{C^k,g^M}(TM)$. We introduce here the transpose of the Levi-Civita connection

$$\nabla(u)^T : TM \to TM \tag{9.9}$$

such that for any $h,k \in T_xM$

$$g_x^M(\nabla(u)h,k) = g_x^M(h,\nabla(u)^T k) \tag{9.10}$$

We introduce the decomposion in symmetric and skew-symmetric part of $\nabla(u)$

$$\nabla(u) = A(u) + S(u)$$

where

$$S(u)h = \frac{1}{2}(\nabla(u)h + \nabla(u)^T h) \text{ and } A(u)h = \frac{1}{2}(\nabla(u)h - \nabla(u)^T h)\,. \tag{9.11}$$

Noticeably, for any $x \in M$, $A(u)_x \in \mathfrak{so}(T_xM)$, i.e. is a infinitesimal rotation of $T_xM$ (for the reference metric $g_x^M$) associated with $u$. Therefore any vector field $u \in \Gamma_{C^{k+1}}(TM)$ defines an element $(u, A(u)) \in \mathfrak{aut}_{C^k,g^M}(TM)$, so that we have the smooth inclusion

$$\Gamma_{C^{k+1}}(TM) \hookrightarrow \mathfrak{aut}_{C^k,g^M}(TM)$$

We will also denote $A(u)$ this element of $\mathfrak{aut}_{C^k,g^M}(TM)$ for simplicity. Let $p \geq 1$ and $I = [0,1]$. Any curve $u_t \in L^p(I,\Gamma_{C^{k+1}}(TM))$ also defines a curve $A(u_t) \in L^p(I,\mathfrak{aut}_{C^k,g^M}(TM))$ and the operator $A$ lifts to a smooth inclusion :

$$L^p(I,\Gamma_{C^{k+1}}(TM)) \overset{A}{\hookrightarrow} L^p(I,\mathfrak{aut}_{C^k,g^M}(TM))$$

Therefore, for $u_t \in L^p(I,\Gamma_{C^{k+1}}(TM))$, we consider the dynamic given by:

$$\frac{D}{dt}\Psi_t = A(u_t)\circ\Psi_t, \quad \Psi_0 = \mathrm{id}_{TM} \tag{9.12}$$

along the curve

$$\partial_t \underline{\Psi}_t = u_t \circ \underline{\Psi}_t, \quad \underline{\Psi}_0 = \mathrm{id}_M \tag{9.13}$$

**Proposition 9.11** (Flow of (9.12)). *Let $u \in L^p(I, \Gamma_{C^{k+1}}(TM))$. Then, the equation* (9.12) *has a unique global (i.e. defined on $I$) solution $\Psi^u \in AC_{L^p}\left(I, \mathrm{Aut}_{C^k,g^M}(TM)\right)$. Moreover, the mapping $u \in L^p(I, \Gamma_{C^{k+1}}(TM)) \mapsto \Psi^u \in AC_{L^p}\left(I, \mathrm{Aut}_{C^k,g^M}(TM)\right)$ is locally bounded, and the restriction $u \in L^p(I, \Gamma_{C^{k+2+l}}(TM)) \mapsto \Psi^u \in AC_{L^p}\left(I, \mathrm{Aut}_{C^k,g^M}(TM)\right)$ with $l \geq 0$ is $C^l$.*

*Proof.* This is a direct consequence of [82, §.2.3] and the fact that $A : L^p(I, \Gamma_{C^{k+1}}(TM)) \to L^p\left(I, \mathfrak{aut}_{C^k,g^M}(TM)\right)$ is smooth. □

**Remark 9.12.** *Not that in the case of automorphisms of $\mathbb{R}^3$, equipped with the standard euclidian metric, the operator $A : \Gamma_{C^{k+1}}(\mathbb{R}^3) \to \mathfrak{aut}_{C^k,g^M}(\mathbb{R}^3)$ is the curl :*

$$\mathrm{curl} : \left\{ \begin{array}{ccc} C^{k+1}(\mathbb{R}^3, \mathbb{R}^3) & \longrightarrow & \mathfrak{aut}_{C^k,g^M}(\mathbb{R}^3) \\ u & \longmapsto & [\partial_y u_z - \partial_z u_y, \partial_z u_x - \partial_x u_z, \partial_x u_y - \partial_y u_z] \end{array} \right.$$

*Therefore, the equation* (9.12) *allows to follow the vorticity*

We will denote $\mathrm{Evol}^A : L^p(I, \Gamma_{C^{k+1}}(TM)) \to AC_{L^p}(I, \mathrm{Aut}_{C^k,g^M}(TM))$ the mapping associating to $u_t \in L^p(I, \Gamma_{C^{k+1}}(TM))$ the associated absolutely continuous curve of automorphisms $\Psi_t$ that is solution of (9.12)

## 9.2 The shape space of $C^k$-metrics and transport of metrics

In this section, we consider the space $\mathrm{Met}_{C^k}(M)$ of Riemannian metrics on the manifold $M$ with $C^k$ coefficients, and we define a natural action of the space of automorphisms of $TM$ on the space of metrics. We study the differentiability of this action and in particular the induced tranport equation induced by the infinitesimal action.

### 9.2.1 Definitions and differentiable structure on $\mathrm{Met}_{C^k}(M)$

We first recall how a Banach differentiable structure can be defined on this space. Let $S_2T^*M$ denote the vector bundle of symmetric $(2,0)$ tensors on $M$, and $S_2^+T^*M$ be the open subspace of definite positive tensors. Then the space of metrics $\mathrm{Met}_{C^k}(M)$ is simply the space of sections $\Gamma_{C^k}(S_2^+T^*M)$ of this fiber bundle. It is therefore endowed with a Banach differentiable structure [80, 31], and it is actually an open subset of the Banach space $\Gamma_{C^k}(S_2T^*M)$, so that

$$T\,\mathrm{Met}_{C^k}(M) = \mathrm{Met}_{C^k}(M) \times \Gamma_{C^k}(S_2T^*M).$$

The group of $C^k$ automorphisms of $TM$ acts naturally on $\mathrm{Met}_{C^k}(M)$ by the pushforward

$$\begin{array}{ccc} \mathrm{Aut}_{C^k}(TM) \times \mathrm{Met}_{C^k}(M) & \longrightarrow & \mathrm{Met}_{C^k}(M) \\ (\Psi, g) & \longmapsto & \Psi_* g \end{array}$$

where for $y \in M$, and $h, k \in T_yM$,

$$(\Psi_* g)_y(h, k) = g_{\underline{\Psi}^{-1}(y)}(\Psi^{-1}k, \Psi^{-1}h).$$

This action satisfies some regularity properties.

**Proposition 9.13** (Regularity of the action)**.** *The action of* $\mathrm{Aut}_{C^k}(TM)$ *on* $\mathrm{Met}_{C^k}(M)$ *is continuous. Moreover for* $l > 0$*, the restriction of the action*

$$\begin{array}{ccc} \mathrm{Aut}_{C^{k+l}}(TM) \times \mathrm{Met}_{C^{k+l}}(M) & \longrightarrow & \mathrm{Met}_{C^k}(M) \\ (\Psi, g) & \longmapsto & \Psi_* g \end{array}$$

*is* $C^l$*.*
*The pullback of metrics*

$$\begin{array}{ccc} \mathrm{Aut}_{C^k}(TM) \times \mathrm{Met}_{C^{k+l}}(M) & \longrightarrow & \mathrm{Met}_{C^k}(M) \\ (\Psi, g) & \longmapsto & \Psi^* g = (\Psi^{-1})_* g \end{array}$$

*is also* $C^l$*.*

*Proof.* Let $l \geq 0$. By introducing local trivialisation $U \subset M$ adapted to $L_1(TM, TM)$ and $S_2 T^* M$, we can reduce to the action of $\mathrm{Aut}_{C^{k+l}}(TU)$ on $\mathrm{M}_{C^{k+l}}(U)$. We introduce the fiberwise pushforward and pullback :

$$P : \left\{ \begin{array}{ccc} GL_n \times S_n^{++} & \longrightarrow & S_n^{++} \\ (\Psi, g) & \longmapsto & \Psi_* g \end{array} \right. , \qquad P' : \left\{ \begin{array}{ccc} GL_n \times S_n^{++} & \longrightarrow & S_n^{++} \\ (\Psi, g) & \longmapsto & \Psi^* g = P(\Psi^{-1}, g) \end{array} \right.$$

where $S_n^{++}$ is the set of positive definite symmetric matrices, and $\Psi_* g \in S_n^{++}$ is defined as $(\Psi_* g)(k, h) = g(\Psi^{-1} k, \Psi^{-1} h)$ for $k, h \in \mathbb{R}^n$. The mappings $P$ and $P'$ are smooth and by [80, theorem 13.5], so are the pushforward $P_*, P'_* : \mathrm{Aut}_{C^{k+l}}(TU) \times \mathrm{Met}_{C^{k+l}}(U) \to \mathrm{Met}_{C^{k+l}}(U)$. Moreover the composition

$$\mathrm{comp} : \left\{ \begin{array}{ccc} C^k(U, U) \times \mathrm{Met}_{C^{k+l}}(U) & \longrightarrow & S_n \\ (\underline{\Psi}, g) & \longmapsto & g \circ \underline{\Psi} \end{array} \right.$$

is $C^l$ and the action of automorphisms on metrics is given by

$$\Psi_* g = C(P_*(\Psi, g), t(\mathrm{inv}(\Psi))$$

is also $C^l$ since the mapping $t \circ \mathrm{inv} : \mathrm{Aut}_{C^{k+l}}(TU) \to \mathrm{Aut}_{C^k}$ is also $C^l$.

Finally, the pullback of metrics by automorphisms is given by

$$\Psi^* g = C(P'_*(\Psi, g), t(\Psi))$$

which is $C^l$ too. □

We denote $\xi$ the infinitesimal action of $\mathfrak{aut}_{C^{k+1}}(TM) \simeq \Gamma_{C^{k+1}}(TM) \oplus \mathrm{End}_{C^k}(TM)$ on $\mathrm{Met}_{C^{k+1}}(M)$, i.e. the derivative of the action:

$$\xi : \left\{ \begin{array}{ccc} \mathfrak{aut}_{C^{k+1}}(TM) \times \mathrm{Met}_{C^{k+1}}(M) & \longrightarrow & T\,\mathrm{Met}_{C^k}(M) \\ ((u, a), g) & \longmapsto & \xi_g(u, a) = \partial_\Psi (\Psi_* g)|_{\Psi = \mathrm{id}_{TM}}.(u, a) \end{array} \right.$$

Note that, for $g \in \mathrm{Met}_{C^{k+1}}(M)$, since the subgroup $\mathrm{Aut}_{C^{k+1}, g}(TM)$ is the stabilizer of $g$, then the kernel of $\xi_g$ is simply the Lie algebra of $\mathrm{Aut}_{C^{k+1}, g}(TM)$:

$$\ker(\xi_g) = \mathfrak{aut}_{C^{k+1}, g}(TM) \simeq \Gamma_{C^{k+1}}(TM) \oplus \mathfrak{so}_{C^{k+1}, g}(TM).$$

We also have the following regularity property of the infinitesimal action.

**Proposition 9.14** (Regularity of the infinitesimal action). *The infinitesimal action*

$$\xi : \mathfrak{aut}_{C^{k+1}}(TM) \times \mathrm{Met}_{C^{k+1}}(M) \to T\,\mathrm{Met}_{C^k}(M)$$

*is smooth vector bundle morphism.*

*Proof.* Since the action on the given space is continuously differentiable, then the infinitesimal action is continuous. Moreover it is the restriction of a bilinear mapping on $\mathfrak{aut}_{C^{k+1}}(TM) \times \Gamma_{C^{k+1}}(S_2T^*M)$, and therefore is smooth □

### 9.2.2 Flows and geometric metric transport

In this part, we assume that the manifold $M$ is equipped with a smooth metric $g^M$, so that $(M, g^M)$ is a smooth compact Riemannian manifold. The metric $g^M$ will be called the **reference metric** and encodes all the metric properties of the ambient space. Therefore, as shapes will be encoded by elements of $\mathrm{Met}_{C^k}(M)$, this will allow to follow the anisotropy of the shapes (with regard to the reference metric $g^M$) through equation (9.12). We start with a formal interpretation of this transport of metrics keeping the anisotropy. We consider a time-dependent $C^{k+1}$-vector field on $M$ given by $(t, x) \to u(t, x)$. We would like to consider the associated flow in the space of metrics generated by $u$ starting from a initial metric $g_0 \in \mathrm{Met}_{C^{k+1}}(M)$. A basic idea would be to consider the flow introduced by the parallel transport of the initial metric along the flow lines i.e. such that $\partial_t g_t + \nabla_{u_t} g_t = 0$. However, we should have an **instantaneous rotation of the metric orientation** associated with the flow allowing the structural anisotropy reorientation of the metric. We propose the following equation :

$$\partial_t g(t) + \nabla_{u(t)} g(t) + A_{u(t)} g(t) = 0, \quad g(0) = g_0 \tag{9.14}$$

where, for $u \in \Gamma_{C^{k+1}}(TM)$, the infinitesimal metric $A_u g \in T_g\,\mathrm{Met}_{C^k}(M)$ is defined by

$$A_u g(h, k) = g(h, A(u)k) + g(A(u)h, k)\,. \tag{9.15}$$

Note that we get from (9.10) that $A_u g^M(a, b) = g^M(a, A(u)b) + g^M(A(u)a, b) = g^M(a, A(u)b) - g^M(a, A(u)b) = 0$, so that $A_u g^M = 0$.

We will prove, under some assumptions, that solutions of this equation can actually be lifted to the half-Lie group $\mathrm{Aut}_{C^k, g^M}(TM)$.

**Proposition 9.15** (Infinitesimal action). *Let $g \in \mathrm{Met}_{C^{k+1}}(M)$, and $u \in \Gamma_{C^{k+2}}(TM)$, so that $A(u) \in \mathfrak{aut}_{C^{k+1}, g^M}(TM)$. Then we get*

$$\xi_g(A(u)) = -\nabla_u g - A_u g \tag{9.16}$$

*Moreover the induced action*

$$\xi^A = \xi \circ A : \begin{cases} \Gamma_{C^{k+2}}(TM) \times \mathrm{Met}_{C^{k+1}}(M) & \longrightarrow & T\,\mathrm{Met}_{C^k}(M) \\ u, g & \longmapsto & -\nabla_u g - A_u g \end{cases}$$

*is a smooth vector bundle morphism*

*Proof.* This follows directly from proposition 9.14 and the smoothness of the operator $A$. □

In particular we get the following result

**Theorem 9.16** (Metric transport equation)**.** *Let $g_0 \in \mathrm{Met}_{C^{k+1}}(M)$ an initial metric, and $u_t \in L^p(I, \Gamma_{C^{k+2}}(TM))$. Then there exists a unique maximal curve $g_t \in AC_{L^p}(I, \mathrm{Met}_{C^k}(M))$ satifying* (9.14)*, i.e. such that $g(t) \in \mathrm{Met}_{C^{k+1}}(M)$ for any $t \in I$ and*

$$\partial_t g(t) + \nabla_{u(t)} g(t) + A_{u(t)} g(t) = 0, \quad g(0) = g_0 \tag{9.17}$$

*Moreover, if $\Psi_t = \mathrm{Evol}^A(u_t) \in AC_{L^p}\left(I, \mathrm{Aut}_{C^{k+1}, g^M}(TM)\right)$ is the unique solution of* (9.12)*, then we get*

$$g(t) = \Psi(t)_* g_0. \tag{9.18}$$

**Remark 9.17.** *We see that the condition that $g(t) \in \mathrm{Met}_{C^{k+1}}(M)$ is necessary in order for the equation* (9.17) *to make sense in* $\mathrm{Met}_{C^k}(M)$. *Indeed the term $\partial_t g(t)$ is in $T\,\mathrm{Met}_{C^k}(M)$ and the operator $-\nabla_u - A_u$ takes a $C^{k+1}$ metric and gives a $C^k$ metric. Moreover, note that even though $g(t) \in \mathrm{Met}_{C^{k+1}}(M)$ for any $t \in I$, the curve $g(t)$ is in general not differentiable in time in the space* $\mathrm{Met}_{C^{k+1}}(M)$*, so that equation* (9.17) *makes sense only in the bigger space* $\mathrm{Met}_{C^k}(M)$. *The solutions of equation* (9.17) *will be here elements of $AC_{L^p}(I, \mathrm{Met}_{C^k}(M)) \cap C^0(I, \mathrm{Met}_{C^{k+1}}(M))$. This particular setting and this loss of regularity for this equation means we will not be able to use the Picard-Lindelöf theorem, and we will have to use the integrated version* (9.18) *to prove existence and uniqueness of solutions.*

*Proof.* We introduce the curve $\Psi_t = \mathrm{Evol}^A(u_t) \in AC_{L^p}\left(I, \mathrm{Aut}_{C^{k+1}, g^M}(TM)\right)$ solution of (9.12). By proposition 9.15, the equation (9.17) can be reformulated as

$$\partial_t g(t) = \xi^A_{g(t)}(u(t)) \tag{9.19}$$

Since the restriction of the action

$$\begin{array}{ccc} \mathrm{Aut}_{C^{k+1}}(TM) \times \mathrm{Met}_{C^{k+1}}(M) & \longrightarrow & \mathrm{Met}_{C^k}(M) \\ (\Psi, g) & \longmapsto & \Psi_* g \end{array}$$

is $C^1$, the curve $t \mapsto \Psi(t)_* g_0 \in \mathrm{Met}_{C^{k+1}}(M)$ is therefore absolutely continuous in $\mathrm{Met}_{C^k}(M)$ and is solution of (9.19).

Conversely, suppose $t \mapsto g(t)$ is an absolutely continuous curve in $\mathrm{Met}_{C^k}(M)$, such that for any $t \in I$, $g(t) \in \mathrm{Met}_{C^{k+1}}(M)$ and $g(t)$ is solution of (9.17). We prove now that for all $t \in I$,

$$\Psi(t)^* g(t) = g_0,$$

i.e.

$$g_t(\Psi_{t,x} h, \Psi_{t,x} k) = g_0(h, k)$$

for any $t \in I$ and $h, k \in T_x M$. Note that the curve $t \mapsto \Psi(t)^* g(t)$ is not necessarily absolutely continuous in $\mathrm{Met}_{C^k}(M)$, since we only have $g(t) \in \mathrm{AC}_{L^p}(I, \mathrm{Met}_{C^k}(M))$. However the evaluation $g_t(\Psi_{t,x} h, \Psi_{t,x} k)$ is absolutely continuous in $\mathbb{R}$ since the mappings

$$\begin{array}{ccc} \mathrm{Aut}_{C^{k+1}}(TM) \times (TM \times_M TM) & \longrightarrow & TM \times_M TM \\ (\Psi, (h, k)) & \longmapsto & (\Psi(h), \Psi(k)) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Met}_{C^k}(M) \times (TM \times_M TM) & \longrightarrow & \mathbb{R} \\ (g, (h', k')) & \longmapsto & g(h', k') \end{array}$$

are $C^1$. We get

$$\begin{aligned}
\partial_t(g_t(\Psi_{t,x}h,\Psi_{t,x}k)) &= (\nabla_{\frac{d}{dt}}g_t)(\Psi_{t,x}h,\Psi_{t,x}k) + g_t\left(\nabla_{\frac{d}{dt}}(\Psi_{t,x}h),\Psi_{t,x}k\right) + g_t\left(\Psi_{t,x}h,\nabla_{\frac{d}{dt}}(\Psi_{t,x}k)\right)\\
&= (\nabla_{\frac{d}{dt}}g_t)(\Psi_{t,x}h,\Psi_{t,x}k) + g_t\left(A(u_t)\Psi_{t,x}h,\Psi_{t,x}k\right) + g_t\left(\Psi_{t,x}h,A(u_t)\Psi_{t,x}k\right)\\
&= (\nabla_{\frac{d}{dt}}g_t)(\Psi_{t,x}h,\Psi_{t,x}k) + (A_{u_t}g_t)(\Psi_{t,x}h,\Psi_{t,x}k)\\
&= (\nabla_{\frac{d}{dt}}g_t + A_{u_t}g_t)(\Psi_{t,x}h,\Psi_{t,x}k) = 0
\end{aligned}$$

Here the last equality follows from the fact that $g_t$ is a solution of (9.17). Therefore for all $t\in I$, $g_t(\Psi_{t,x}h,\Psi_{t,x}k) = g_0(\Psi_{0,x}h,\Psi_{0,x}k) = g_0(h,k)$, and we get the result. $\square$

We finish this section by studying flows of conformal metrics with regards to the reference metric $g^M$.

**Proposition 9.18.** *Let $u_t\in AC_{L^p}(I,\Gamma_{C^{k+2}}(TM))$ and suppose the initial metric $g_0 = \alpha g^M$ is conformal. Then the solution $t\to g_t$ of (9.17) stays conformal at any time and satisfy*

$$g_t = \alpha\circ\underline{\Psi}^{-1}g^M$$

*Proof.* Indeed, since by definition of the Levi-Civita connection, we have $\nabla_u g^M = 0$ and $\nabla_u(\alpha_t g^M) = (\nabla_u\alpha_t)g^M$ where $\alpha_t = \alpha\circ\Psi_t^{-1}$. We just have to check that $\partial_t\alpha_t + \nabla_u\alpha_t = 0$ which is true since we have $\alpha_{t+s}(\underline{\Psi}_{t+s}(x)) = \alpha(x)$ for $s\geq 0$. We conclude by uniqueness property of the solution. $\square$

When the initial metric $g_0$ is not conformal, the evolution is more complex. However, we still have the important property that the density of $\mathrm{Vol}(g_t)$ with respect to $\mathrm{Vol}(g_M)$ is advected along the flow. Here $|d\phi_t|$ play has a **growth factor in Lagrangian coordinate**.

### 9.2.3 Advection of the density along the flow

Here we consider the density $\rho_t$ such that

$$\mathrm{Vol}(g_t) = \rho_t\,\mathrm{Vol}(g^M) \tag{9.20}$$

**Theorem 9.19.** *For any arbitrary initial condition, we have $\frac{D}{dt}\mathrm{Vol}(g_t) = 0$ and $\rho_t\circ\varphi_t = \rho_0$.*

*Proof.* Indeed, we have $\frac{D\,\mathrm{Vol}(g_t)}{dt} = \frac{1}{2}\mathrm{Vol}(g_t)\,\mathrm{tr}(g^{-1}\frac{Dg}{dt})$. From (9.17), we get

$$\mathrm{tr}(g^{-1}\frac{Dg}{dt}) = -\,\mathrm{tr}(g^{-1}A_u g) = \mathrm{tr}(g^{-1}(gA(u)+A(u)^Tg)) = \mathrm{tr}(A(u)) - \mathrm{tr}(A(u)) = 0$$

so that $\frac{D}{dt}\mathrm{Vol}(g_t) = 0$ from which $\rho_t\circ\varphi_t = \rho_0$ follows immediately. $\square$

We deduce from the theorem that for any measurable domain $\Omega_0\subset M$, if $\Omega_t = \varphi_t(\Omega_0)$ we have

$$\int_{\Omega_t}\mathrm{Vol}(g_t) = \int_{\Omega_t}\rho_t\,\mathrm{Vol}(g_M) = \int_{\Omega_0}\rho_t\circ\varphi_t|d\varphi_t|\,\mathrm{Vol}(g^M) = \int_{\Omega_0}\rho_0|d\varphi_t|\,\mathrm{Vol}(g^M) \tag{9.21}$$

which is the usual $d$-varifold action, that we often consider when modeling tissues. Note that this is not the usual advection equation for mass preserving transport.

## 9.3 Pullback of Ebin metric

In the previous sections 9.1 and 9.2, we introduced a differential setting that allows to study transport of metrics. In particular, this allows us to explore new variational problems and in particular to study here the role of anisotropy. As an example, we introduce in this section a variational problem for the matching of metrics, using a pullback of the Ebin metric. This metric was first introduced in [31] and widely studied due to its numerous applications, in particular in optimal transports. We refer to [37, 24] for general results on this metric, and in particular about the metric completion of the space of Riemannian metrics with regards to the Ebin metric. We then propose a series of simple numerical examples, using a triangle discretization.

### 9.3.1 Definition of the metric and first properties

We consider for any $u \in \Gamma_{C^k}(TM)$ and $g \in \mathrm{Met}_{C^k}(M)$ the cost function

$$c(g,u) = G_g(S_u g, S_u g) = \int_M \mathrm{tr}(g^{-1}(S_u g)g^{-1}(S_u g))\,\mathrm{Vol}(g) \tag{9.22}$$

where $S_u g \in T_g \mathrm{Met}_{C^k}(M)$ is defined by

$$S_u g(a,g) \doteq g(S(u)a,b) + g(a, S(u)b)\,.$$

It is important to note that $S_u : \mathrm{Met}_{C^k}(M) \to T\,\mathrm{Met}_{C^k}(M)$ is a linear operator that depend on $g^M$. In particular, we also see that

$$-S_u g = \xi^A_g u - \xi^{\mathrm{Diff}}_g u$$

where $\xi^{\mathrm{Diff}}_g u$ is the infinitesimal action of $u$ on $g$ induced by the pushforward action of the group $\mathrm{Diff}_{C^k}(M)$ on the space of metrics :

$$\xi^{\mathrm{Diff}}_g u = \partial_\varphi(\varphi^* g)|_{\varphi=\mathrm{id}} u$$

Note that we use here the notation $\xi^{\mathrm{Diff}}_g$ to highlight it is the infinitesimal action. But we can also see that we get $\xi^{\mathrm{Diff}}_g u = -\mathcal{L}_u g$ where $\mathcal{L}_u g$ denotes the classic Lie derivative of the metric $g$ given by

$$\mathcal{L}_u g(h,k) = \nabla_u g(h,k) + g(\nabla_h u, k) + g(h, \nabla_k u)$$

Since we are studying the dynamic of (9.14), we see we are not evaluating here the energy of the time derivative $\dot{g}(t)$, but we are isolating the contribution of the action of automorphisms removing the contribution of the action of the diffeomorphisms. Moreover, this cost is also vanishing along the flow of isometries of $g^M$ :

**Proposition 9.20** (Vanishing property of (9.22))**.** *Assume that $u_t \in L^p([0,1], \Gamma_{C^{k+2}}(TM))$ and let $g_t$ and $\Psi_t$ be solutions of* (9.14) *and* (9.12)*. If $\int_0^1 c(g_t, u_t)dt = 0$, then for any $t \in [0,1]$, $\Psi_t = d\varphi_t$ and $\varphi_t$ is* ***an isometry*** *from $(M, g_0)$ to $(M, g_t)$ and from $(M, g^M)$ to $(M, g^M)$.*

*Proof.* We have that $c(g_t, u_t) = 0$, meaning that $S(u_t) = 0$ so that $\frac{D}{dt} d\varphi_t = \nabla(u) \circ d\varphi_t = A(u) \circ d\varphi_t$. From theorem 9.16, we deduce that $\Psi_t = d\varphi_t$ and the first isometry property follows immediatly from theorem 9.16, and in particular equation (9.18). The second one follows from proposition 9.18 from $g_0 = g^M$. □

From proposition 9.20, we get that if the cost is vanishing along the flow, then whatever $g_0$ is, the resulting flow $\varphi_t$ is in the isometry group of $M$ equipped with reference metric $g^M$. In standard situation, this group is pretty small. In the case $\mathbb{R}^d$ with the standard euclidian metric, we get the euclidian group and positively oriented affine rotation. On the sphere $S^d$, we get the restrictions of $SO_{d+1}$.

### 9.3.2 Variational problem

Now suppose we are given a source metric $g_S \in \mathrm{Met}_{C^k}(M)$ and a target $g_T \in \mathrm{Met}_{C^k}(M)$. We also consider a dissimilarity term $\mathcal{D} : \mathrm{Met}_{C^k}(M) \to \mathbb{R}$, which allows to define the inexact matching problem

$$\inf_{(u)\in L^2([0,1],\Gamma_{C^k}(TM))} J(u) = \frac{1}{2}\int_0^1 c(g_t, u_t)\,dt + \mathcal{D}(g_1) \tag{9.23}$$
$$\text{s.t} \quad \begin{cases} \partial_t g(t) = \xi^A_{g(t)} u(t) = -\nabla_{u(t)} g(t) - A_{u(t)} g(t) \\ g(0) = g_S \end{cases}$$

In this setting, one might measure the distance between the final metric $g_1$ and the target metric $g_T$ by the Frobenius norm of their difference. In practice, however, the numerical experiments we will implement rely on a weaker observation model: only the Riemannian volume form is assumed to be observable. This motivates an attachment based on the associated measures.

$$\mathcal{D}(g_1) = \tilde{\mathcal{D}}(\mathrm{Vol}(g_1))$$

### 9.3.3 Discretization and numerical examples

We propose in the following section some simple numerical experiments to illustrate this framework. As a discretization of the continuous setting, we encode metrics with triangulations of the domain with constant metrics on each triangle. We take the ambient space $M$ to be included in $\mathbb{R}^2$, although this choice departs from the compact framework. A natural compact alternative would be to embed the domain into the flat torus $\mathbb{T}^2$, which admits compatible triangulations. The action of automorphisms is then applied only to this finite collection of triangles.

#### 9.3.3.1 Triangulated model

The triangulation is denoted

$$\mathcal{T} = (V, F),$$

with $V \subset \mathbb{R}^2$ a finite set of vertices and $F$ a set of oriented triangular faces. A discrete Riemannian metric is thus represented by a collection of metrics on triangles

$$g = (g_T)_{T\in F}, \qquad g_T \in S_2^{++},$$

where $S_2^{++}$ is the space of symmetric positive definite matrices. Each $g_T$ encodes the local metric structure on the triangle $T$.

Moreover, in this discretization, we will suppose presence of triangles carrying zero density, and thus having no contribution to the Ebin energy. We will therefore restrict to the subdomain of $M$ of triangles with positive mass, allowing this subdomain to be transported by the action of the control field $u$. Moreover, this model also extends the theoretical pushforward action: the acting object is no longer a group of automorphisms but rather a groupoid of local automorphisms of $TM$ restricted to subdomains.

#### 9.3.3.2 Discretization of controls.

Vector fields $u$ are encoded as continuous piecewise affine functions, affine on each triangle and continuous across edges. Their Jacobians are therefore piecewise constant. In the flat torus case, this construction leads to a finite-dimensional differential structure associated with a groupoid. Therefore, in this setting, the dynamic (9.14), governed by a control $u_t$ is described by

$$\begin{cases} \dot{x}_t = u_t(x_t), & \forall x_t \in V_t \\ \dot{g}_{T_t} = A(u_t)g_{T_t} - g_{T_t}A(u_t), & \forall T_t \in F_t \end{cases}$$

where $(V_t, F_t)$ denotes the triangulation at time $t$, and $A(u) = \frac{1}{2}(\nabla u - \nabla u^\top)$ is the skew-symmetric part of the Jacobian of $u$.

In particular the discrete setting does not coincide with the smooth action of $\mathrm{Aut}_{C^k}(TM)$ on $\mathrm{Met}_{C^{k+1}}(M)$, but should be viewed as a finite-dimensional approximation that converges to the continuous model in the limit of mesh refinement.

#### 9.3.3.3 Variational Model

The discrete framework is equipped with a variational formulation (9.23) driving the deformation between two triangulated meshes. The central ingredient is the Ebin metric, adapted to the discretized setting, together with a data attachment functional.

**Attachment term.** As we stated previously, our numerical experiments rely on a weak observation model: only the Riemannian volume form is assumed to be observable. This motivates an attachment based on the associated measures. Each mesh $g$ induces a discrete measure

$$\mu_g = \sum_{T \in F} \mu_T \, \delta_{c_T},$$

where $c_T$ is the barycenter of a triangle $T$ and $\mu_T = |T|\sqrt{\det g_T}$ is its integrated volume form. Given two meshes $g$ and $g'$, their attachment energy is then defined as the squared RKHS norm

$$\mathcal{D}(g, h) = \|\mu_g - \mu_h\|^2_{V_\sigma^*},$$

where $V_\sigma$ is the Gaussian RKHS associated to a scale $\sigma$.

**Ebin regularization.** For the regularization, we discretize the Ebin metric on the space of triangulated metrics. In presence a vector field control $u$ we get the following form of the Ebin energy:

$$E_{\mathrm{Ebin}}(g, u) = \sum_{T \in F} \mu_T \,\mathrm{tr}\left( (g_T^{-1}(S_T g_T + g_T S_T))^2 \right),$$

where $S_T$ denotes the symmetric part of $\nabla u$ restricted to the triangle $T$, and $\mu_T$ is the integrated volume form.

**Energy functional.** Suppose we are given an initial mesh metric $g_S$ and a target mesh metric $g_T$. The variational problem (9.23) in this discretized setting now becomes the following optimization problem over the time-dependent control $u_t$

$$J(u) = \lambda \int_0^1 E_{\text{Ebin}}(g_t, u_t)\, dt + E_{\text{att}}(g(1), g^1),$$

where $g(t)$ is the mesh metric transported from $g^0$ under the controls, and $\lambda > 0$ balances fidelity and regularity. The trajectory $(g(t))$ is obtained by time integration of the discrete dynamics (pushforward by infinitesimal rotations, and vertex transport), and the minimization is carried out by quasi-Newton methods.

#### 9.3.3.4 Numerical examples

**Anisotropic annulus optimization** To illustrate the behavior of our metric-based deformation model under anisotropic changes, we consider a simple two-region geometry composed of a disk with concentric annular structure. Each annulus is endowed with a piecewise constant Riemannian metric, defined separately for the inner disk and the outer ring.

Figure 9.1 shows three template configurations:

1. an isotropic metric (identity) on the whole domain,
2. an anisotropic inner disk (with a vertical stretching),
3. an anisotropic inner disk rotated by 45° (tilted anisotropy).

The target geometry is generated by an incompressible affine transformation with eigenvalues $\alpha$ and $-\alpha$, corresponding to horizontal stretching combined with horizontal contraction.

We optimize the deformation from each template to the target using the full Ebin energy. The results are shown in Figure 9.2, and the discretization we described in the previous sections. The first observation is that all three templates successfully reproduce the global outer shape of the target. For the inner disk, however, the outcome depends strongly on anisotropy: the isotropic and vertically anisotropic templates produce almost identical reconstructions, whereas the tilted anisotropic template leaves the inner disk essentially undeformed.

These differences can be explained as follows. There is no density contrast between the inner and outer regions, and the data attachment term used here is insensitive to the local anisotropy of the inner metric relative to the outer one. Moreover, the experiment involves only spatial transport, without conformal action, and the chosen "copy-and-paste" action preserves local density in Lagrangian coordinates. Under these assumptions, isotropic and vertically anisotropic templates behave almost identically with respect to the target. By contrast, the tilted anisotropic case is fundamentally different. Indeed, the target was generated by an incompressible affine mapping aligned with the horizontal and vertical axes, which coincide with the principal directions of the vertical anisotropic source. This suggests that any velocity field $u$ with strain tensor $S(u) = (\nabla u + \nabla u^\top)/2$ aligned with these axes ($\theta = 0$) has the same local Ebin energy in both isotropic and vertically anisotropic settings. In the tilted case ($\theta = \pi/4$), the misalignment these two orthogonal frames maximizes the energetic penalty.

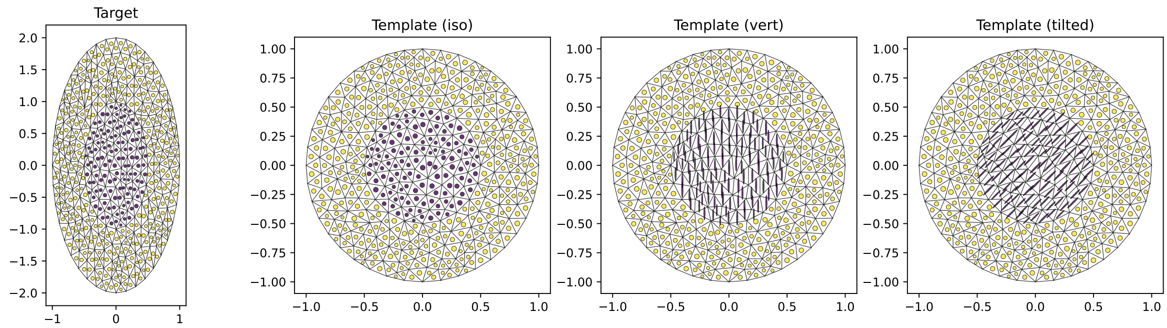


Figure 9.1: Three template metrics and the target geometry before optimization. Left to right: target, isotropic, vertical anisotropy and tilted anisotropy. Each disk illustrates the unit ball of $g_T^{-1}$ in each triangle.

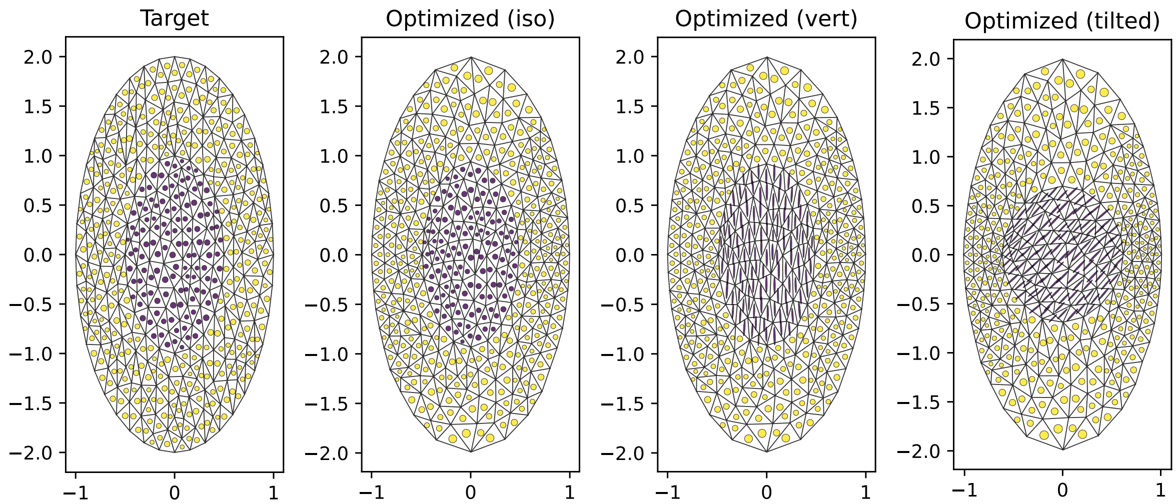


Figure 9.2: Result of the optimization from each template to the target using full Ebin energy. Left to right: target, optimized isotropic, vertical followed by the tilted anisotropy.

**Star-shaped Templates with Anisotropic Cores** We consider here another example with an initial star-shaped mesh, equipped with three different core metrics:

1. isotropic
2. vertically anisotropic
3. tilted anisotropic

The target is defined through an incompressible affine deformation. Figure 9.3 shows the three template meshes with their anisotropic cores, visualized through metric disks. The common target is displayed in Figure 9.4.

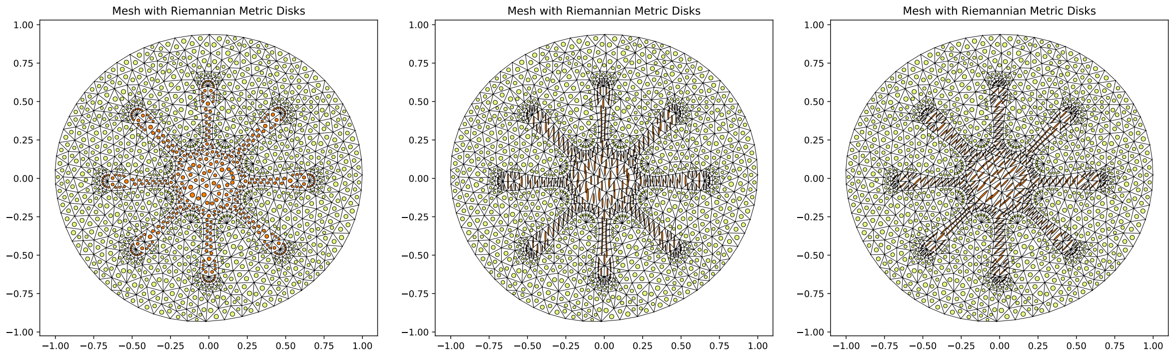


Figure 9.3: Star templates with isotropic core (left), vertical anisotropy (middle), and tilted anisotropy (right).

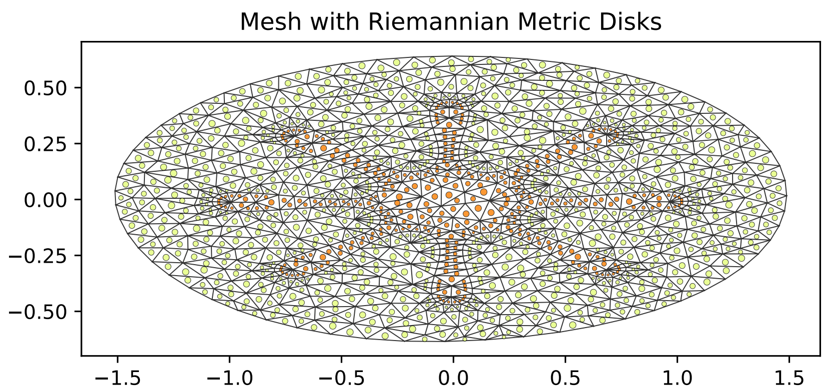


Figure 9.4: Affine-incompressible target for the star experiment.

The deformation towards the affine target is obtained by minimizing the Ebin-penalized attachment energy defined in 9.3.3.3 in two successive phases. Figure 9.5 display the refined states for the three modalities. We observe in particular the same phenomena as in the previous experiment : in the case of a vertically anisotropic metric, the affine stretching deformation is aligned with the horizontal and vertical axes, and therefore with the vertical frames given by the metrics within the star. As a result, the vertical and horizontal branches of the stars follow the stretching deformations closely. Conversely, when the star is equipped with titled anisotropic metrics, the deformation becomes more difficult and requires higher energy, leading to very limited displacements.

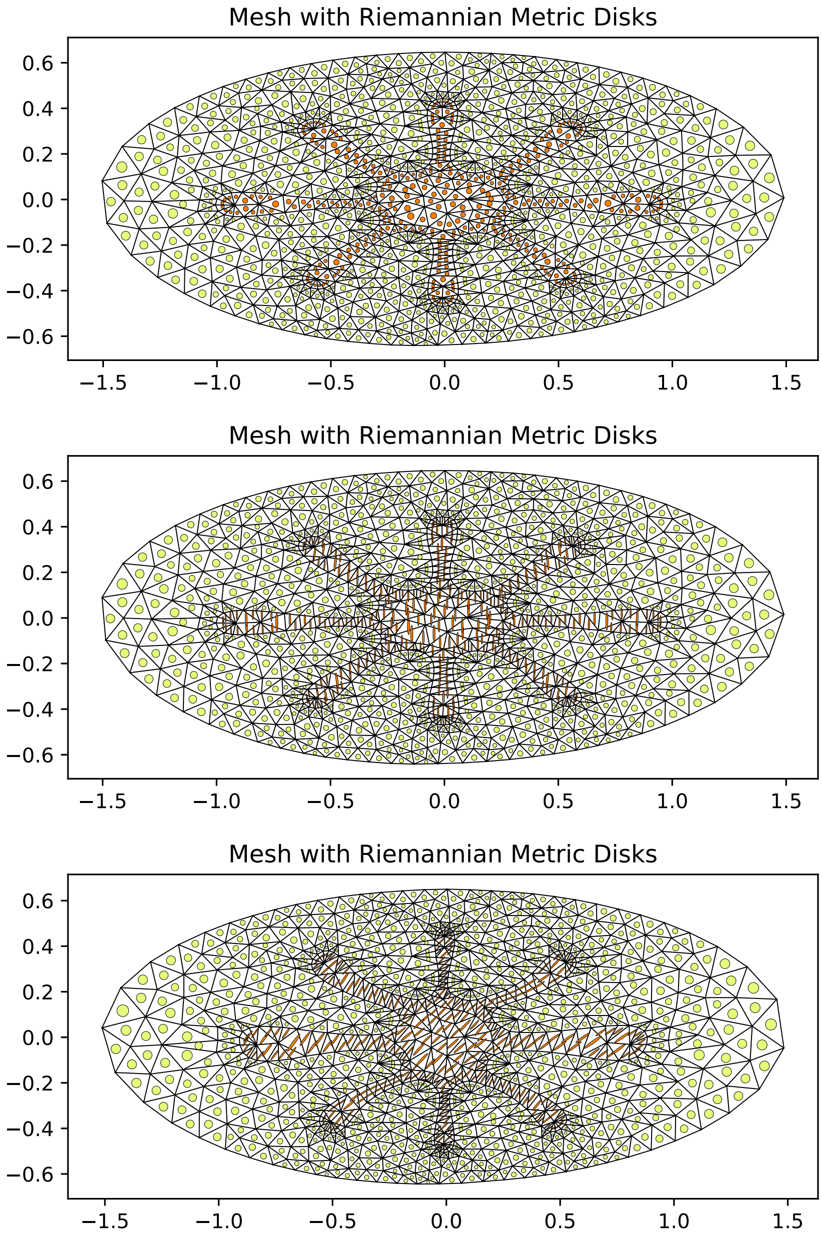


Figure 9.5: Refined deformed meshes for the three templates.

**Discussion** These experiments illustrate the fact that this model only penalizes misalignment between the frames given by the metrics on each triangle and the direction induced by the displacement from the control $u$. In other words, the model does not really account for the eigenvalues of the metrics, which represent the weights along their principal axes; it mainly considers the mass and the orthogonal frame instead. This suggests that, in this model, the metric may encode more information than what is actually exploited in the energy, and that the shape's anisotropy could instead be represented solely through local frames and masses.

# Chapter 10

# Conclusion

## Contents



In this thesis, we presented a geometric framework that extends large deformation model. We first studied Banach half-Lie groups, which are topological groups and Banach manifolds such that the group structure and the differential structure are partially compatible. We proved, under some suitable assumptions, a $L^p$-regularity property on such groups. At the same time, we also endowed the space of absolutely continuous curves in $G$ with a Banach differential structure.

This first results allowed us to study strong right-invariant sub-Riemannian structures on Banach half-Lie groups, and in particular we managed to retrieve completeness results. We also characterized the normal geodesics of these structures, and we related them to the Euler-Arnold Poincaré equation on the dual of the tangent space at identity.

Building on the framework of [6], we extended the definition of shape spaces by considering them as Banach manifolds acted upon by a half-Lie group endowed with a strong sub-Riemannian metric. This allowed us to induce these metrics in the shape spaces, and to formulate a variational problem for inexact matching of shapes. We then gave some applications involving multiscale matching and the integration of anisotropic information into the dynamics.

However, this work focused mainly on theoretical foundations for large deformation models in shape analysis, motivated by applications. We did not dwell too much on the questions raised by the discretization and the implementation of these new variational problems, which would require further developments and extensive studies.

Our framework opens several perspectives, which we discuss next in terms of possible future directions.

## 10.1 Hierarchical multiscale scheme and optimization problems

We come back to the multiscale setting, re-introducing the inexact matching problem

$$\inf_{v^1,v^2} \frac{1}{2}\int_0^1 \left|v_t^1\right|_{V_1}^2 + \left|v_t^2\right|_{V_2}^2 dt + \mathcal{D}(\varphi_1^1, \varphi_1^2) \tag{10.1}$$

where $V_1, V_2 \subset C_0^2$ are RKHS induced by isotropic Gaussian kernels, with scale parameters $\sigma_1 < \sigma_2$ , and $\varphi_1^1, \varphi_1^2$ given by the equations

$$\begin{cases} \varphi_0^1, \varphi_0^2 = \mathrm{id} \\ \dot{\varphi}_t^1 = v_t^1 \circ \varphi_t^1 \\ \dot{\varphi}_t^2 = (v_t^1 + v_t^2) \circ \varphi_t^2 \end{cases}$$

Note that the diffeomorphism $\varphi^1$ represents the coarse-scale deformation, and $\varphi^2$ the deformation at the fine scale. If we consider the usual action on the space of multi-scale landmarks $(\mathbb{R}^d)^{k_1} \times (\mathbb{R}^d)^{k_2}$, we then reduce the problem to minimizing the following functional

$$L(p_i^1, p_j^2) = H(q_i^1, q_j^2, p_i^1, p_j^2) + \mathcal{D}(q_1^1, q_2^2) \tag{10.2}$$

with $H$ the associated Hamiltonian given by

$$H(q_i^1, q_j^2, p_i^1, p_j^2) = \frac{\alpha_1}{2}\left\| K_{\sigma_1}\left(\sum_i \delta_{q_i^1}^{p_i^1} + \sum_j \delta_{q_j^2}^{p_j^2}\right)\right\|_{V_1}^2 + \frac{\alpha_2}{2}\left\| K_{\sigma_2}\sum_j \delta_{q_j^2}^{p_j^2}\right\|_{V_2}^2$$

The challenge is then to choose the parameters $\alpha_1, \alpha_2, \sigma_1, \sigma_2$ and to propose conditions on the $p_i^1, p_j^2$ in the optimization for matching problems. A first naive approach for the optimization would be to perform a 2-steps procedure, starting by fixing $p_j^2 = 0$ and optimizing over the variables $p_i^1$ corresponding to the coarse deformation. A good choice of parameters would then ensure that the second optimization does not "unlearn" the first, by exploiting for example the symmetries and the invariance in the Hamiltonian.

## 10.2 Transport of metrics

In section 9.3, we introduced a variational matching problem using the Ebin metric on the space of metrics. We discuss other possible directions.

### 10.2.1 Existence and characterization of minimizers

In section 9.3, we introduced the following variational problem for metric matching :

$$\inf_{(u)\in L^2([0,1],\Gamma_{C^k}(TM))} J(u) = \frac{1}{2}\int_0^1 c(g_t, u_t)\, dt + \mathcal{D}(g_1)$$

$$\text{s.t} \quad \begin{cases} \partial_t g(t) = \xi_{g(t)}^A u(t) = -\nabla_{u(t)} g(t) - A_{u(t)} g(t) \\ g(0) = g_S \end{cases}$$

In particular, we have not addressed the question of the existence of minimizers, and the characterization of the critical points. Note that this problem involves a sub-Riemannian structure that is neither strong nor right-invariant, so there might not be a nice geodesic equation associated with this problem. However, thanks to the differential setting we introduced in sections 9.1 and 9.2, results from optimal control could be used to establish the existence of minimizers.

### 10.2.2 Exploring other variational problems

In section 9.3, we explored a first variational model based on the $L^2$-Ebin metric. As seen in section 9.3.3, this model exhibits interesting phenomena, and penalizes misalignment between orthogonal frames given by the metrics on the triangles and orthogonal frames given by the control $u$. However, the actual weights given by the eigenvalues of each metric are not considered in this model. It may therefore be desirable to define energies that penalize control directions associated with low eigenvalues. This will be subject to future investigations.

### 10.2.3 Allowing conformal changes

A wider framework can be considered with addition of a new action given by conformal rescaling of the metric leading to something akin to the so called metamorphosis framework [94, 35]. Let us consider a curve $(u_t, \eta_t)$ in $\Gamma_{C^k}(TM) \times C^k(M, \mathbb{R}_{>0})$, and the associated integration path $(\Psi, \beta)$ starting from $(\mathrm{id}_M, 1_M)$ and solution of

$$\begin{cases} \partial_t \underline{\Psi}_t = u_t \circ \varphi_t \\ \frac{D}{dt}\Psi_t = A(u_t) \circ \Psi_t - \eta_t \circ \varphi_t \Psi_t \\ \partial_t \beta_t = 2(\eta_t \circ \varphi_t)\beta_t \end{cases} \tag{10.3}$$

Thus the new equation on the metric $g_t = \Psi_{t*} g_0$ now becomes

$$\partial_t g_t + \nabla_{u_t} g_t + A_{u_t} g_t = 2\eta_t g_t \tag{10.4}$$

which thus allows conformal changes of the metric. Following the metamorphosis framework, we can update the previouns functional (9.23) adding a cost for the conformal factor

$$c(g, u) = G_g(S(u) + 2\eta g, S(u) + 2\eta g) = \int_M \mathrm{tr}(g^{-1}(S_u g + 2\eta g) g^{-1}(S_u g + 2\eta g)) \,\mathrm{Vol}(g)$$

### 10.2.4 Discretization and half-Lie groupoid actions

In sections 9.1 and 9.3, we introduced a differential framework for the continuous model of Riemannian metric transport, which allowed us to formulate a variational problem for metric matching. Yet, the discretization raises further questions regarding the differential framework for the discretized transport of metrics. Indeed, in this discrete setting, metrics are now encoded on triangles, which implies that the action we define must respect the underlying triangulation structure. This naturally gives rise to a groupoid of deformations, where the set of objects is the set of all triangulations with metrics defined on each face of the triangles, and the set of morphisms between objects is the set of automorphisms preserving these structures. In such case, this framework still lies within the setting of infinite dimensional geometry, and this groupoid turns out to be a half-Lie groupoid, in the sense the multiplication and the inverse are not smooth. Similarly, the space of controls associated with this discretized framework is then a (half)-Lie algebroid. Many questions are thus raised from this perspective, in particular regarding the differential structures of such objects and how this discrete setting can be related to the continuous framework as the number of triangles tends to infinity.

## 10.3 Towards a hybrid framework of intrinsic and extrinsic metrics

The first part of this manuscript focused on right-invariant metrics on shape spaces, that is, extrinsic metrics induced by deformations of the ambient space that do not depend on the shapes themselves. However, we extended this framework in chapters 8 and 9, where anisotropic features of the shapes play a role in the definition of the metric on the shape space. These examples give rise to new sub-Riemannian structures that are not right-invariant and can be formulated as a combination of extrinsic, LDDMM-like metrics and intrinsic metrics depending on the shapes. Therefore, hybridizing intrinsic and extrinsic metrics provides a richer and more flexible framework for modeling deformations in shape analysis allowing to encode the geometric properties of the shapes themselves. This hybrid formulation opens to new theoretical questions and challenges in numerical modeling (cf. also [22] for similar ideas)

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